ALGEBRA

with Applications to Business and Economics

1239

This book is in the

ADDISON-WESLEY SERIES IN MATHEMATICS

ALGEBRA

with Applications to Business and Economics

by

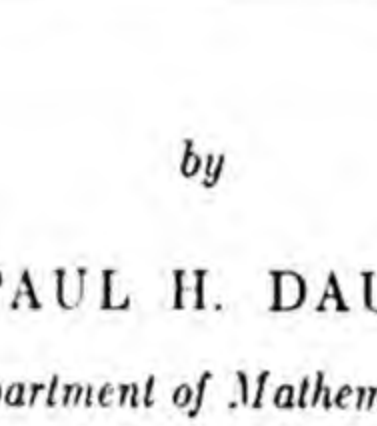
PAUL H. DAUS

Department of Mathematics
University of California, Los Angeles

and

WILLIAM M. WHYBURN

Department of Mathematics
University of North Carolina



ADDISON-WESLEY PUBLISHING COMPANY, INC.

READING, MASSACHUSETTS · PALO ALTO · LONDON

THIS BOOK IS AN ADDISON-WESLEY
WORLD STUDENT SERIES EDITION

A complete and unabridged reprint of the original American textbook, this World Student Series edition may be sold only in those countries to which it is consigned by Addison-Wesley or by its authorized Trade Distributors, Japan Publications Trading Company, Ltd., Tokyo, or Book Centre, Ltd., London. It may not be re-exported from the country to which it has been consigned, and it may not be sold in the United States of America or its possessions.

Cat.
512
D263A

ACM
1398

ADDISON-WESLEY PUBLISHING COMPANY, INC.

ALL RIGHTS RESERVED. THIS BOOK, OR PARTS THEREOF, MAY NOT BE REPRODUCED IN ANY FORM WITHOUT WRITTEN PERMISSION OF THE PUBLISHERS.

Library of Congress Catalog Card No. 61–10565

PREFACE

This text is designed with the following objectives in mind: first, to provide manipulative skills which are needed for the successful pursuit of texts like the authors' *Introduction to Mathematical Analysis with Applications to Problems of Economics;* second, to introduce the student to certain ideas and terminology which are useful at the level of intermediate algebra or intermediate college algebra; third, to provide applications of algebraic material to business and economics.

There are a number of existing books that partially meet these objectives, but there is no single text which includes all the material that the authors consider important for successful pursuit of a course which uses their earlier text, cited above. As was indicated in the Preface of that text, an introduction to vector and matrix algebra, and the very important topic of linear programming, are highly desirable. Such topics necessitate the inclusion and development of the subject of linear inequalities far beyond that given in standard texts on intermediate algebra or intermediate college algebra. This also reflects the modern trend to include linear equalities, well developed, in the second course in algebra given in high schools, as well as in special freshman courses designed for college students of the social sciences. Although manipulative skills are essential and important, the present text is prepared from the point of view that ideas and understanding of mathematical concepts are even more important.

The text may contain more material than can be covered in a three-hour, one-semester course. This is especially true for those students who have had only the required minimum of mathematical training in high school. Such students need at least a four-hour, one-semester course, or the text material should be carefully selected. For these students, Chapters 7, 8, and parts of Chapter 10 may be omitted. For better qualified students, Chapters 1, 2, 3, and parts of Chapter 4 contain review material, while a more mature treatment is presented in Chapter 7. It is intended that this text, followed by our *Introduction to Mathematical Analysis with Applications to Problems of Economics*, will provide a complete year course for students in schools of business administration. The present book purposely contains some materials included in the other, and the use of both texts permits certain topics to be reviewed, or the total material to be compressed so that both texts can be completed in one academic year.

The authors express their appreciation to users of their text *Introduction to Mathematical Analysis with Applications to Problems of Economics* for many suggestions that have helped in the preparation of the present book. Staff members of Addison-Wesley have given all possible assistance as the manuscript was prepared, edited, and brought to publication. We record our gratitude to them and express a hope that the book will justify a large portion, at least, of the effort and care that they have given to it.

A Suggested Semester Course Outline

This outline is based upon the assumption that the class meets four days a week for the first four weeks (longer if necessary) and after a placement test is given, the meetings are reduced to three a week, except that the weaker students meet on the extra day for review and drill.

The outline takes into consideration the fact that Chapters 1, 2, 3, and part of Chapter 4 are review work, and that Chapters 1, 2, 3, 4, 5, 6, and 9 are essential for the next course. Chapters 7 and 8 contain some of the more interesting and important parts of algebra and its recent applications to business analysis.

Each problem set contains many more problems than can ordinarily be assigned. The average assignment should include from one-half to one-third of the problems, representing different ideas where possible.

Assignment No.	Section	Problem Set
1	1–1, 1–2	1–1
2	1–3 to 1–6	1–2
3	1–7 to 1–9	1–3, 1–4
4	2–1 to 2–6	2–1, 2–2
5	3–1 to 3–5	3–1, 3–2
6	3–6, 3–7	3–3
7	3–8	3–4
8	3–9	3–5
9, 10	3–10, 3–11	3–6, 3–7
11	4–1, 4–2	4–1
12	4–3	4–2
13	4–4	4–2, 4–3
14	4–5	4–4
15	4–6	4–5
16	4–7	4–6
17	Placement Test (see above)	
18	5–1 to 5–4	5–1, part 5–2
19	5–5	5–2
20	5–6	5–3
21, 22	5–7, 5–8	5–4
23	6–1 to 6–3	6–1
24, 25	6–4 to 6–6	6–2
26	6–7	6–3
27	6–8	6–4
28	Test	

For the rest of the course some choice must be made, but the order of the topics is not important.

Assignment No.	Section	Problem Set
29	9–1 to 9–3	9–1
30	9–4, 9–5	9–2
31, 32	9–6	9–3
33, 34	9–7	9–4
35	7–1 to 7–4	Read only; omit 7–1 to 7–3
36, 37	7–5 to 7–7	7–4
38	7–8, 7–9	7–5
39	8–1, part 8–2	8–1, part 8–2
40	8–2	8–2
41	8–3	8–3
42	8–4, 8–5	8–4
43	8–6	8–5
44 Test		

Select one of the following alternatives:

I 45	8–7	8–6
I 46, 47	8–8	8–7
I 48, 49	8–9	8–8
II 45	10–1, 10–2	10–1
II 46	10–3, 10–4 (omit proof)	10–2
II 47, 48	10–5, 10–6	10–3
II 49	10–7 (selection)	10–4

If more time is available or the class is amply prepared, include the omitted parts of Chapters 5, 6, 7, and the alternate assignments of parts of Chapters 8 and 10. This will extend over ten to twelve additional assignments.

P.H.D.
W.M.W.

CONTENTS

CHAPTER 1. THE REAL NUMBER SYSTEM. ADDITION AND SUBTRACTION 1

1–1 The real number system 1
1–2 Symbols. Numbers 2
1–3 Equality 6
1–4 Addition of real numbers 7
1–5 Subtraction 9
1–6 Positive and negative numbers 10
1–7 The number scale and linear measure 12
1–8 Rules of sign. Removal of parentheses 14
1–9 Multiplication by an integer 18

CHAPTER 2. MULTIPLICATION 22

2–1 Multiplication of real numbers 22
2–2 Multiplication by zero 23
2–3 Rules of sign 25
2–4 Removal of parentheses 27
2–5 Positive integral exponents 28
2–6 Multiplication of polynomials. Special products 29

CHAPTER 3. DIVISION AND FRACTIONS 32

3–1 Reciprocal of a number 32
3–2 Division as the inverse of multiplication 33
3–3 Rules of sign 34
3–4 Equivalent fractions (change of form) 34
3–5 Algebra of fractions 36
3–6 Division as successive subtraction. Remainder 39
3–7 Decimal fractions. Definition and notation 40
3–8 Percentage and applications 48
3–9 Factoring 52
3–10 Division of polynomials 56
3–11 Factoring polynomials 59

CHAPTER 4. LINEAR EQUATIONS 64

4–1 Equations 64
4–2 Linear equations in one variable 67
4–3 Worded problems and linear equations 70
4–4 Equivalent rates. Successive discounts 77
4–5 Simple interest and simple discount 80
4–6 Linear functions and linear equations in two variables 84
4–7 Applications 90

CHAPTER 5. SIMULTANEOUS LINEAR EQUATIONS 97

5-1 Simultaneous linear equations in two variables 97
5-2 Algebraic solution 97
5-3 Graphical solution. Dependent and inconsistent equations . . 100
5-4 Equation of a line determined by two conditions 103
5-5 Worded problems and simultaneous linear equations 104
5-6 Market equilibrium with linear demand and supply laws . . 108
5-7 Determinants of the third order 111
5-8 Simultaneous linear equations in three variables 115
5-9 Graphical interpretation 119

CHAPTER 6. QUADRATIC EQUATIONS 129

6-1 Square root 129
6-2 Decimal approximation to $\sqrt{n}$ 130
6-3 Algebraic operations with square roots 131
6-4 Algebraic solution of a quadratic equation 133
6-5 Complex numbers 136
6-6 Equations involving square roots. 136
6-7 The quadratic function and its graph 141
6-8 Applications 147
6-9 Circle and ellipse 151
6-10 Hyperbolas 153
6-11 Intersection of a line with simple conics 158

CHAPTER 7. LAWS OF ALGEBRA. EQUALITY AND INEQUALITY . . . 163

7-1 Introduction 163
7-2 Equality 164
7-3 Addition and subtraction 164
7-4 Multiplication 167
7-5 Order and inequality 171
7-6 Algebraic operations with inequality 172
7-7 Linear inequalities 175
7-8 Quadratic inequalities 180
7-9 Completing the real number system 184

CHAPTER 8. INTRODUCTION TO MATRIX ALGEBRA AND LINEAR PROGRAMMING 187

8-1 Row vectors and column vectors 187
8-2 Geometric interpretation of vectors 192
8-3 Matrices 199
8-4 Vector multiplication 204
8-5 Matrix multiplication 207
8-6 Properties of matrix multiplication 211
8-7 Matrices and linear equations 217
8-8 Linear inequalities in two and three variables 226
8-9 Linear programming 234

CHAPTER 9. EXPONENTS, RADICALS, AND LOGARITHMS 245
9–1 Positive and negative integral exponents 245
9–2 Rational exponents and radicals 246
9–3 Irrational exponents 248
9–4 Logarithms 251
9–5 Tables of common logarithms 253
9–6 Numerical calculations 256
9–7 Compound interest 261

CHAPTER 10. MATHEMATICAL INDUCTION, PROGRESSIONS, BINOMIAL THEOREM, AND ANNUITIES 269
10–1 Mathematical induction 269
10–2 Arithmetic progression 271
10–3 Geometric progression 276
10–4 Binomial theorem 281
10–5 Annuity formulas 284
10–6 Annuity tables 286
10–7 Special annuity problems 290

APPENDIX I. Equation of a plane 301

APPENDIX II. Division process for $\sqrt{n}$ 305

APPENDIX III. Tables 307
I. Squares and square roots 309
II. Common logarithms 310
III. Accumulated amount of 1 312
IV. Present value of 1 316
V. Accumulated value of annuity of 1 per period 320
VI. Present value of annuity of 1 per period 324

ANSWERS TO PROBLEMS 329

INDEX 349

CHAPTER 1

THE REAL NUMBER SYSTEM. ADDITION AND SUBTRACTION

1–1 The real number system. We shall start our study of the real number system by first discussing familiar types of real numbers, and will defer our formal discussion of this concept until Chapter 7. The *natural numbers*, also referred to as the *positive integers* or the *counting numbers*, one,. two, three, four, . . . , are represented by the special symbols 1, 2, 3, 4, ... Each natural number except the first is obtained by the successive addition of 1 to its predecessor: 1, $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$, and so on. With each such natural number there is associated a word and a symbol. At the start, these words and symbols are simple, but they soon become compound words and symbols, such as twenty-three and 23.

The set of all numbers of the type just described is called the set of positive integers. When two numbers of this set are added or multiplied, the result is always another positive integer. However, when one number of the set is divided by another, the quotient is not necessarily a positive integer. It is convenient, then, to enlarge the concept of numbers to include the quotient of any two positive integers, such as $\frac{2}{3}$, $\frac{4}{5}$, $\frac{8}{3}$. This extension of the number concept is made in such a way that the existing laws of operation do not have to be changed and, insofar as possible, in such a way that the new numbers—with appropriate interpretations—obey these same fundamental laws of operation. Indeed, this is the desired goal whenever the number concept is extended, although this objective cannot always be achieved.

The set of all numbers that can be obtained by the division of one natural number by another natural number is called the set of (*positive*) *rational numbers*. This set contains all of the common fractions, such as those indicated above. Terminating decimal fractions, such as 0.23, 2.37, 0.00248, are included, since these may be written as common fractions with denominators which are powers of 10. There are numbers that are not rational: for example, $\sqrt{2}$, $\sqrt{5}$, and the special number pi (π), which is the ratio of the circumference of a circle to its diameter. Such numbers are called *irrational numbers*. It can be proved that when a rational number is expressed as a decimal fraction, the decimal equivalent either terminates or repeats. For example, $\frac{3}{16} = 0.1875$ terminates, whereas $\frac{3}{7} = 0.428571\,428571\ldots$ repeats. Conversely, terminating or

repeating decimal fractions always correspond to rational numbers. The decimal equivalents of $\sqrt{2}$, $\sqrt{5}$, and π neither terminate nor repeat.

There is one other extension of the number system which plays an essential part in algebra, and this enlarges the real number system to include the number zero and negative numbers. Although no formal definition of a real number has been given at this point, it is suggested that the totality of positive and negative, terminating and infinite, decimal fractions be thought of whenever the real number system is used. This set includes: (1) the set of *natural numbers*; (2) the set of *integers*, which by definition contains the natural numbers, zero, and the negative integers; (3) the set of *rational numbers*, where we may include the negative rational numbers as well as the positive rational numbers; and (4) the *irrational numbers*.

In addition to the real numbers, there are complex numbers, such as $4 + \sqrt{-3}$. Also, the number concept will be further extended in Chapter 8 to include certain sets of real numbers as special numbers. Some of the usual laws of algebra do not hold for these numbers which do not belong to the real number system.

1–2 Symbols. Numbers. This section discusses the use of symbols to represent (a) numbers or sets of numbers, (b) operations with numbers, and (c) relations between numbers.

(a) *Use of symbols to represent numbers or sets of numbers.* In arithmetic, some real numbers are represented by special symbols or combinations of simple symbols, such as $1, 2, 3, \ldots, 23, \ldots, 302, \ldots; \frac{2}{3}, 0.74; \sqrt{2}, \sqrt[3]{5}, \pi$. Special symbols do not exist for all real numbers. One of the features of algebra which distinguishes it from arithmetic is its use of letters to represent numbers or sets of numbers. Since this representation is not unique, its use must be accompanied by some statement, called the *legend*, about these numbers. Illustrations are:

(1) If a and b are natural numbers, . . . ;
(2) If a and b are real numbers, . . . ;
(3) Find the (real) number x such that . . . ;
(4) Let Z be the set of all natural numbers: $Z = \{1, 2, 3, \ldots\}$.
(5) Let $\mathcal{R}$ be the set of all real numbers.

The symbols used in arithmetic represent definite fixed values. Inasmuch as the letters used in algebra may have several meanings, the legend must specify how they are to be interpreted.

A *constant* is a quantity which remains fixed throughout a given discussion. Thus the symbols used in arithmetic to represent numbers of fixed values denote constants; the letters a and b used in illustrations (1) and (2) above denote constants, for their values remain fixed throughout

a given discussion. It is customary to use the first letters of the alphabet: a, b, c, d, . . . for constants. If the letters a, b, c, d, . . . are used and no legend is stated, it is understood that they are constants.

A *variable* is a quantity which may take on, one at a time, values of constants in a given set. The given set, called the domain of the variable, is specified by some legend. If z may take on the natural numbers of the set Z of illustration (4) above, and if r may take on the real numbers of the set $\mathfrak{R}$ of illustration (5), then z and r are variables—the domain of z being the set of natural numbers and the domain of r being the set of real numbers. This book follows the convention that if no domain is stipulated or implied, the domain is the set of all real numbers. Illustrations:

(6) For any x, $x^2 - 4 = (x - 2)(x + 2)$.
It is understood that "any" refers to any real number.

(7) If y is defined by the equation $y = 6 - 3x$, it is understood that the domain of x is the set of real numbers. If the further conditions are imposed that both x and y are greater than or equal* to zero ($x \geqq 0$, $y \geqq 0$), then x cannot exceed 2, so that the domain of x is the set of numbers consisting of 0, 2, and all real numbers between 0 and 2.

An *unknown* is a variable whose constant values are to be determined by conditions or imperative legends placed on it. A common form of these imperative legends is: "Find x such that . . . ," where the three dots represent some condition imposed on the variable. Illustrations:

(8) Find x such that $3x = 10$.
The set of numbers to be determined consists of the single constant $\frac{10}{3}$.

(9) Find a positive integer x such that $3x \geqq 12$ and $3x \leqq 20$.
The set of numbers to be determined consists of the constants 4, 5, 6.

(10) Find x and y such that $y = x + 1$ and $x + y = 5$.
It is understood that x and y are both unknowns. It is not difficult to show that $x = 2$, $y = 3$ is the single pair of constants determined by the given conditions.

(11) Find real numbers x and y such that $x + y \leqq 5$, $-x + y \geqq 1$, and $3x + 2y$ is as large as possible. The solution set for the unknowns x and y is the single pair of constants $x = 2$, $y = 3$.

The connection between "variable" and "unknown" is close, and it is customary to represent either of them by the letters at the end of the alphabet: x, y, z. This convention is not always followed, and frequently the initial letter of a key word is used instead of x, y, or z. If the letters x, y, z are used, and no statement is made about them, it is usually under-

* For a brief discussion of inequalities, see the end of this and the next section, and for a precise discussion, see Sections 7-5 and 7-6.

stood that they are variables whose domains are the set of real numbers. In general, a variable is referred to as an unknown when some imperative statement is given which requires determination of those constant values which satisfy one or more conditions. Illustration:

(12) The simple interest formula is $I = PRT$, where P represents the principal, R the interest rate, T the time, and I the amount of interest. Any one of these four symbols may be considered a variable, but a real understanding of the physical situation shows that P and I are positive real numbers; T is a positive real number, usually less than 1; and R is a positive real number with a limited range. If specific values are assigned to the symbols P, R, T and equation (12) is accompanied by an instruction to find I, then I is the unknown. If P and T are assigned specific values and R is permitted to take any real positive value less than or equal to 10%, then R is a variable whose domain is specified as $0 < R \leqq 10\%$. I might also be considered as a variable whose domain is to be computed. This is called the *range* of I and is given by $0 < I \leqq PT/10$.

(b) *Use of symbols to represent operations with numbers.* The usual operations of elementary arithmetic are well known, and these will receive additional emphasis as the nature of the real number system and its use in algebra and analysis are discussed. Special symbols are used to represent these operations: addition by $+$, subtraction by $-$, multiplication by $\times$ or a center dot $\cdot$. When numbers are represented by letters, multiplication is often indicated by placing the letters in juxtaposition, ab meaning $a \times b$. The latter notation is not appropriate when the symbols represent the natural numbers—since a symbol such as 34 would not be understood as 3×4 but would mean "thirty-four." The operation of division is also indicated by several symbols: $\div$; a fraction bar, $-$; a solidus, /. Thus, if a and b are real numbers,

$$a \div b, \quad \frac{a}{b}, \quad a/b$$

all mean the same thing. The operation of extracting a square root of a number is indicated by the special symbol $\sqrt{\ }$. Other special symbols are used as needed when new operations are introduced.

(c) *Use of symbols to represent relations between numbers.* Two or more numbers, constants, or variables may be connected in some relationship which is specified by words or by conventional symbols. The most common such relationships are denoted by the symbols for equality, $=$; less than, $<$; and greater than, $>$. Equality is discussed in the next section, where the laws concerning this relation will be given. The other two symbols, $\leqq$ and $\geqq$, will be mentioned briefly, but the complete and precise discussion of the laws concerning these relations is deferred until Sections 7–5

and 7-6. The functional relationship between two variables will be discussed in Chapter 4, especially Section 4-6.

A combination of symbols which represent numbers and operations is called an *algebraic expression*. A relation between two algebraic expressions is called a *formula*. A formula, together with its appropriate legend, is called an *equation* (not necessarily an equality). Sometimes the legend is not stated explicitly but is implied by the context.

In an equation which involves an unknown, the stated condition may be true for some values of the unknown and false for other values. If the statement is true for some constant, that constant is called a *solution* of the equation. The totality of all such solutions is called the *solution set* for the equation.

EXAMPLES:

(1) $x(a + b)$, $\sqrt{a - b}$, $(ax + by)/(2x - 3y)$ are *algebraic expressions.*
(2) $I = PRT$, $y = 6 - 3x$, $x \geqq 0$, $A = \pi r^2$ are examples of *formulas.*
(3) Examples of equations are:

Find x such that $3x = 10$ (the solution set is the single number $\frac{10}{3}$); find the positive integers such that $12 \leqq 3x \leqq 20$ (the solution set consists of the integers 4, 5, 6); given P, R, T and $I = PRT$, find I; for any x, $x^2 - 4 = (x - 2)(x + 2)$ (here the solution set is the set of all real numbers). Without appropriate legends, the various equations or inequalities stated above are formulas.

PROBLEM SET 1-1

1. Numbers may be real, complex, rational, irrational, integral, nonintegral, natural, negative, zero, or positive. A number may belong to several of these categories. Classify each of the following numbers.

$$1,\quad 13,\quad -11,\quad 0,\quad \tfrac{3}{4},\quad \tfrac{6}{4},\quad \tfrac{8}{4},\quad -\tfrac{11}{15},\quad 0.25,\quad 0.333,\quad \sqrt{2},\quad \sqrt[3]{5},\quad \pi,$$

$$\tfrac{22}{7},\quad 3.1416,\quad -\frac{2\pi}{3},\quad \sqrt{-2},\quad \sqrt{-4},\quad \sqrt[3]{-2},\quad \sqrt[3]{-8}.$$

2. Give at least one other example of each type of number listed in problem 1.
3. Classify the following decimal fractions using the types listed in problem 1:
   (a) the repeating infinite decimal 0.333 . . . ,
   (b) the repeating infinite decimal 0.909090 . . . ,
   (c) the infinite decimal 0.12345678910111̇2 . . . , formed by writing in order the symbols which give the natural numbers in order.
   (d) the infinite decimal 0.10 110 1110 . . . , where the law of formation is: write one 1, then one 0, two 1's, then a 0, three 1's, then a zero, and so on, increasing the number of 1's by one each time.
4. Give at least one other example of the types represented by the numbers listed in problem 3.

5. Give some symbols which represent constants and some which represent variables. If possible, explain the meaning of the symbol.

6. Write some formulas (from geometry, physics, or business) and explain what each symbol stands for. Explain under what conditions each symbol is thought of as a variable and give its domain. Also explain under what conditions the symbol might be considered as an unknown.

7. In the formulas given for problem 6, explain which of the symbols stand for operations and which stand for relations between numbers.

8. Give the symbol for a "relation" other than $=$, $<$, $>$. Give meaning to your symbol and discuss some of its properties.

1–3 Equality. Two real numbers are equal if they have the same value. The form of the numbers may be different, that is, they may have different names or be given in terms of different symbols. To say that they are equal implies that they could be reduced to the same form, although this reduction might not always be simple. When $a = b$ is written, it means that a and b are different names for the same numbers but that forms of a and b may be very different. For example:

$$2 = \tfrac{4}{2} = \sqrt{4} = 1.999\ldots \text{(forever)},$$

and

$$\tfrac{1}{8} + \tfrac{1}{8} = \tfrac{2}{8} = \tfrac{1}{4} = 0.25,$$

$$\sqrt[3]{26 + 15\sqrt{3}} + \sqrt[3]{26 - 15\sqrt{3}} = 4.$$

The symbol for equality, $=$, is always used in the above sense, even when the entities are not real numbers. If a and b are symbols representing objects (in algebra the objects may be numbers; in geometry the objects may be lines or other geometric entities), the equation $a = b$ means that a and b are different names for the same object. With this interpretation the following laws concerning equality seem intuitive, but they are really assumptions. In Chapter 7, they appear as the Axioms of Equality. In Chapters 1 and 2, where no attempt is made to prove the statements, they are called laws or rules.

LAWS OF EQUALITY. *If a, b and c are real numbers, then*

E1. THE REFLEXIVE LAW.

$a = a$.

E2. THE SYMMETRIC LAW.

If $a = b$, *then* $b = a$.

E3. THE TRANSITIVE LAW.

If $a = b$ *and* $b = c$, *then* $a = c$.

E4. THE RULE OF SUBSTITUTION.

If a occurs in any algebraic expression and if $a = b$, then b may be substituted for a to give an algebraic expression which is equal to the original expression.

The foregoing is equivalent to saying that the name or form of the expression may be changed by the indicated substitution but not its value. This substitution may also be applied to any formula (a relation between algebraic expressions), and the resulting formulas continue to express the same idea that the original one did.

The statement $a = a$ is given as a law to assert that it is a true relationship which involves the real number a and the undefined relation of equality. There are other relations represented by symbols, such as greater than, $>$, where the relation $a > a$ is not a true statement. Inclusion of the reflexive law for equality gives one way to distinguish the equality relation from other relations for which the reflexive law is not assumed.

The relation $a > 0$ is used to mean "a is positive." In the early chapters, the inequality concepts are used on an intuitive basis, while in Chapter 7 precise formulations of the axioms and theorems concerning inequalities are given.

1–4 Addition of real numbers. The laws of addition listed below are stated so that the *operation of addition* has the familiar properties which it has for natural numbers or for those extensions with which the reader is already familiar. Statement of these laws for real numbers is one step in the evolutionary process of developing the real number system. These laws are not independent of each other, since some of them can be proved as theorems on a basis of the others. Here they are stated as laws without proof and are used in the development of manipulative skills of algebra as extensions of the manipulative skills of arithmetic.

A1. CLOSURE.

If a and b are real numbers, there is a unique real number $a + b$, called their sum.*

A2. THE COMMUTATIVE LAW.

For any real numbers a and b,

$$a + b = b + a.$$

A3. THE ASSOCIATIVE LAW.

For any real numbers a, b, c,

$$(a + b) + c = a + (b + c) = a + b + c.$$

---

* The word "unique" is used throughout this text to mean "one and only one."

The parentheses are used to indicate that the operation within them is to be performed first. The law states that the same result is obtained irrespective of the order in which the additions are made. This is then represented as $a + b + c$ without the use of parentheses.

If laws A2 and A3 are applied to any finite set of real numbers, the result is stated in this general form:

The order in which real numbers are added does not affect their sum.

The symbol 0, for zero, is used in elementary arithmetic as a placeholder in the positional notation for writing numbers. In algebra, the number zero is a most important real number and requires special attention because of its character. It is sometimes referred to as the *additive identity* and is defined by the following assumption:

A4. ZERO.

For any real number a, there exists a unique real number, called zero and represented by the symbol 0, such that

$$a + 0 = a.$$

In view of the commutative law for all real numbers, the law could have been written in the more general form:

$$a + 0 = 0 + a = a.$$

A5. ADDITIVE INVERSE.

For each real number a there exists a unique real number x such that

$$a + x = 0.$$

This number x is called the *additive inverse of a* and is represented by prefixing a negative sign, $-$, before the symbol a and enclosing both symbols in parentheses: $(-a)$. With this notation, the law A5 can be stated in this form:

For each real number a, there exists a unique real number $(-a)$ such that

$$a + (-a) = 0.$$

An important observation to make, in view of the commutative law of addition, is that a is also the additive inverse of $(-a)$. Thus:

$$(-(-a)) = a. \tag{1-1}$$

The additive inverse of the additive inverse of a given real number is the number itself.

The number $(-a)$ is also called the *negative* of the number a, and *if a is a positive number* (Section 1-6), *$(-a)$ is called a negative number.* There

is no *a priori* method for knowing whether a given letter represents a positive or a negative number. If the natural numbers are assumed to be positive numbers, then their additive inverses or negatives: (-1), (-2), (-3), etc., are negative numbers.

A6. ADDITION OF EQUALS.

For any real numbers a, b, c, d, *if* $a = b$, *then* $a + c = b + c$.

If $c = d$, then the rule of substitution and the transitive law of equality show that *if* $a = b$ *and* $c = d$, *then* $a + c = b + d$.

This law is often stated in the form:

If equals are added to equals, the sums are equal.

A7. CANCELLATION FOR ADDITION.

For any real numbers a, b, c, *if* $a + c = b + c$, *then* $a = b$.

1–5 Subtraction. Subtraction is the inverse of the operation of addition, in the sense that subtraction undoes what addition does. Addition of 3 to 6 gives 9, and subtraction of 3 from 9 yields 6.

DEFINITION OF SUBTRACTION. If x, a, and b are real numbers such that $x + a = b$, then x is the result of subtracting a from b. The operation of subtraction is represented by the minus sign, $-$.

$$x + a = b \quad \text{means} \quad x = b - a. \tag{1-2}$$

THEOREM 1–1. *For any pair of real numbers* a *and* b, *there is a unique real number* x *such that*

$$x + a = b, \quad \textit{namely}, \quad x = b + (-a).$$

Proof. $b + (-a)$ is a solution of the equation* because

$$[b + (-a)] + a = b + [(-a) + a] = b + 0 = b.$$

Use has been made of the associative law of addition A3, the basic properties of the additive inverse A5, and the property of zero A4.

The equation can have only one solution. Suppose that y is also a solution of the equation, so that $x + a = b$ and $y + a = b$. Then the transitive law of equality E3 yields $x + a = y + a$, and the law of cancellation

---

* When we use two signs of aggregation, it is understood that the innermost operation is performed first. Thus $[b + (-a)]$ means: first form the additive inverse of a, then add it to b; this sum is then added to a.

for addition A7 shows $x = y$. Hence there is only one solution, and the proof of the theorem is complete.

Because $x = b + (-a)$ is the only solution of $x + a = b$, and the operation of subtraction was defined so that $x = b - a$, we have

COROLLARY 1-1. $$b - a = b + (-a). \tag{1-3}$$

This property justifies use of the symbol $-$ to represent both the operation of subtraction and the formation of the negative of a number. To distinguish between these two operations, parentheses have been used to represent the negative of a number, but when no misunderstanding can occur, the parentheses are omitted and $-a$ is written instead of $(-a)$, $-(-a)$ is written instead of $(-(-a))$.

1-6 Positive and negative numbers. The natural numbers have been referred to as the positive integers, and it has been noted that the sum of two positive integers is a positive integer (see the discussion after A1, Section 1-4). This concept extends to real numbers, and extension is made at various places in this book. For the present, the following laws concerning positive numbers are stated:

P1. *The number one* (1) *is positive.*

P2. *The sum of two positive numbers is a positive number.*

The natural numbers are obtained by successive additions of 1, and it follows from P1 and P2 that the natural numbers are all positive. This justifies the designation of the natural numbers as positive integers. The assumption P2 applies to any two positive numbers, whether or not they are positive integers. If, for example, $\sqrt{2}$ and π are positive numbers, then the real number $\sqrt{2} + \pi$ is also positive.

DEFINITIONS.

(1) If a is a positive number, then its additive inverse, $(-a)$, is a *negative number.*

(2) The additive inverse of any positive integer is a *negative integer.*

(3) The symbol for greater than, $>$, is used in such way that $a > 0$ means *a is positive.*

(4) The symbol for less than, $<$, is used so that $a < 0$ means *a is negative.*

That is, every positive number is greater than zero and every negative number is less than zero.* Furthermore, if $a > 0$, then $(-a) < 0$.

* For a more complete and precise discussion, see Sections 7-5 and 7-6.

P3. LAW OF TRICHOTOMY.

Every real number is either positive, zero, or negative. That is, if a is any real number, then one and only one of the following relations is true:

$$a > 0, \qquad a = 0, \qquad a < 0.$$

If $b - a$ is positive, $(b - a) > 0$, then $b > a$; and if $b - a$ is negative, $(b - a) < 0$, then $b < a$. When a and b are different positive integers, then the addition facts of arithmetic determine whether $b - a$ is a positive integer or a negative integer (Section 1–8), and hence whether $(b - a) > 0$ or $(b - a) < 0$.

DEFINITION. The *absolute value* of a real number a is represented by the symbol $|a|$ and is defined as follows:

if a is positive $(a > 0)$, then $|a| = a$;
if $a = 0$, then $|a| = 0$;
if a is negative $(a < 0)$, then $|a| = (-a)$.

From Definition (1) above, if a is negative, then $a = (-p)$, where p is a positive number. Since $(-a) = (-(-p)) = p$ by Eq. (1–1), it follows that the absolute value of a nonzero number is always positive.

$$|5| = 5; \qquad |-5| = 5; \qquad |b - a| = \begin{cases} b - a, & \text{if } b > a, \\ a - b, & \text{if } b < a. \end{cases}$$

PROBLEM SET 1–2

1. From the following set of numbers, pick out those that are equal to -2, to $\frac{2}{3}$, to π.

$\frac{3}{2}$, $\frac{4}{6}$, $-\frac{8}{4}$, $\sqrt{2}$, $\sqrt{4}$, $\sqrt[3]{-8}$, $-\frac{14}{7}$, $\frac{22}{7}$; 3.1416, 0.666 . . . forever.

2. If $a = 6$, $b = 2$, $x = 5$, $y = 3$, determine the values of the following algebraic expressions.

(a) $a + x + y$, $a + xy$, $2a - 3b + 4y$

(b) $\dfrac{ax + by}{3x - 2y}$, $\dfrac{a + 2x - 3y}{b + 2x - 3y}$, $\dfrac{ab + xy}{ax + by}$

(c) $\sqrt{a - b}$, $\sqrt{a - y}$, $\sqrt{a + 2b + xy}$

3. Are the following sets of numbers closed under addition? That is, is the sum of two numbers of the set always a number of the set?

(a) the even positive integers
(b) the odd positive integers
(c) the integers (positive, zero, and negative)

(d) the integers with zero omitted
(e) the set consisting of the three numbers:

$$\{-3, 0, 3\}; \qquad \{-a, 0, a\} \quad \text{for any real } a.$$

4. Are the sets listed in problem 3 closed under subtraction?

5. Use the associative law of addition and the following definitions of 2, 3, 4 to prove $2 + 2 = 4$. What other laws were used in this proof?

Definitions: $1 + 1 = 2$; $2 + 1 = 3$; $3 + 1 = 4$.

6. What additional definition, along with those given in problem 5, would you need to prove $2 + 3 = 5$? Give such a definition and prove $2 + 3 = 5$.

7. For any a and b, the equation $x + a = b$ has a unique solution. Give the solution for each of the following.

(a) $x + 3 = 8$ (b) $x + (-3) = 8$
(c) $x + 8 = 3$ (d) $x + (-8) = (-3)$

8. Do the following equations have solutions? Is each solution unique? Explain.

(a) $x + x = x$ (b) $x + x = 2x$ (c) $x + x = 3x$
(d) $2 + x = x$ (e) $2 + x = 2$ (f) $a + x = x$
(g) $a + x = a$

9. Give the two subtraction facts that correspond to the equations.

(a) $a + b = c$ (b) $a + 0 = a$

10. (a) Show that the equation $|a| + |b| = |a + b|$ is satisfied by the numbers $a = 8, b = 2$ and also by the numbers $a = -8, b = -2$. Suggest the general law. (b) Show that the equation $|a| + |b| > |a + b|$ is satisfied by the numbers $a = 8, b = -2$ and also by the numbers $a = -8, b = 2$. Suggest the general law.

1-7 The number scale and linear measure. It is convenient to have a pictorial representation of the real number system. One familiar such representation is a *marked ruler* or *scale*. The usual one-foot ruler, marked in inches, bears the numbers 1, 2, 3, . . . , 12. The interval between two of these marks is further subdivided into eighths and sixteenths, or else into tenths to correspond to decimal fractions. Further subdivisions are estimated by eye. The beginning of the scale represents zero, whether or not it is so marked. Instead of an inch scale, we may have a centimeter scale, such as is illustrated in Fig. 1-1, without further subdivisions.

The scale might be extended and marked as in Fig. 1-2.

FIGURE 1-1

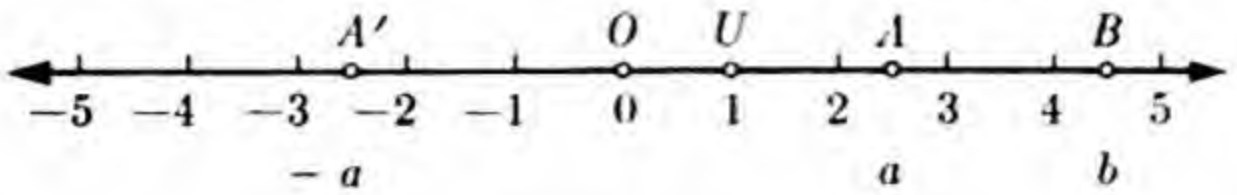


FIGURE 1–2

Such an extended scale is now described more formally. We start with a line of unbounded extent and assume the following:

AXIOM OF LINEAR MEASURE. *The points on a line can be put into one-to-one correspondence with the real numbers such that*

(1) *two different points O and U may be assigned the numbers* 0 *and* 1, *respectively;*

(2) *if the points A and B correspond to the numbers a and b, respectively, then the directed distance from A to B, represented by* $d(AB)$, *is given by* $d(AB) = b - a$.

DEFINITIONS.

(1) The number corresponding to a point is called the *coordinate* of the point. If the point is labeled with a capital letter, it is convenient to represent the coordinate by the corresponding lower-case letter, and this is indicated by writing $A(a)$, $B(b)$, $C(c)$, etc.
(2) The *length* AB, represented by the symbol AB, is defined as $|b - a|$.
(3) Two directed distances $d(AB)$ and $d(PQ)$, with the coordinates given by a, b, p, q, respectively, have the *same directed distance* if and only if

$$b - a = p - q;$$

and the *length* AB *and length* PQ are equal if and only if

$$|b - a| = |p - q|.$$

The foregoing equation is true if either

$$b - a = p - q \quad \text{or} \quad b - a = q - p.$$

Certain elementary characteristics of the number scale are: (a) The given line may be in any position in the plane. (b) The points O and U may be selected arbitrarily so that the unit of measure is arbitrary. (c) The directed distance between two successive natural numbers (positive integers) is always $1 = d(OU)$.

Suppose that A' is the reflection of the point $A(a)$ on the point O, that is, suppose $d(A'O) = d(OA)$ (Fig. 1–2). Then the coordinate of A' is $(-a)$. To prove this statement, let the coordinate of A' be x. Then $d(A'O) = d(OA)$ is equivalent to $0 - x = a - 0 = a$, since $a - 0 = a$ (see

problem 9(b), Problem Set 1–2). The definition of subtraction and the existence of the additive inverse, A5, imply that $a + x = 0$ and $x = (-a)$. This result shows the geometric significance of the relation between positive and negative numbers.

If A is reflected on O to obtain A', and A' is then reflected on O, the point A is obtained. This corresponds to the algebraic fact that

$$-(-a) = a.$$

If a is positive, $a > 0$, then $d(OA) = a > 0$ and the point A and the point U are in the same direction from O; if $a < 0$, the reflection property indicates that the point A and the point U are in opposite directions from O. If $b > a$, then the direction from A to B is the same as that from O to U; and if $a > b$, the direction from A to B is the opposite of that from O to U. Usually, when the line is drawn horizontally, one selects the point U to the right of O; and if the line is vertical, the point U is selected above O, but these selections are arbitrary and not always convenient.

DEFINITIONS.

(1) Let $A(a)$, $B(b)$, $C(c)$ be fixed points on a given line, and let $X(x)$, $Y(y)$ be variable points on the line. If $X(x)$ is such that either $a < x < b$ or $a > x > b$, the point X is *between* A and B, and the set of all such points X, together with the *endpoints* A and B, is called the *segment* $\overline{AB}$. Note that $\overline{AB}$ is used to represent a geometric entity, while AB is used to represent the length of $\overline{AB}$, a positive real number. This definition could be written as

$$\overline{AB} = \{X \mid a \leqq x \leqq b \quad \text{or} \quad a \geqq x \geqq b\},$$

which is read: "the segment $\overline{AB}$ is the set of all points X such that the coordinates satisfy either $a \leqq x \leqq b$ or $a \geqq x \geqq b$.

(2) The set of points X such that $x \geqq a$ and the set of points Y such that $y \leqq a$ are called *rays*. If $c < a < b$, the first ray is represented by $\overrightarrow{AB}$ and the second by $\overrightarrow{AC}$. In other words:

$$\text{ray } \overrightarrow{AB} = \{X \mid x \geqq a, \quad \text{and where } b > a\},$$

$$\text{ray } \overrightarrow{AC} = \{Y \mid y \leqq a, \quad \text{and where } c < a\}.$$

1–8 Rules of sign. Removal of parentheses. When several operations are involved in an algebraic expression, parentheses and other signs of aggregation are used to indicate which operation should be performed first. When the operations are those of addition and subtraction, the fol-

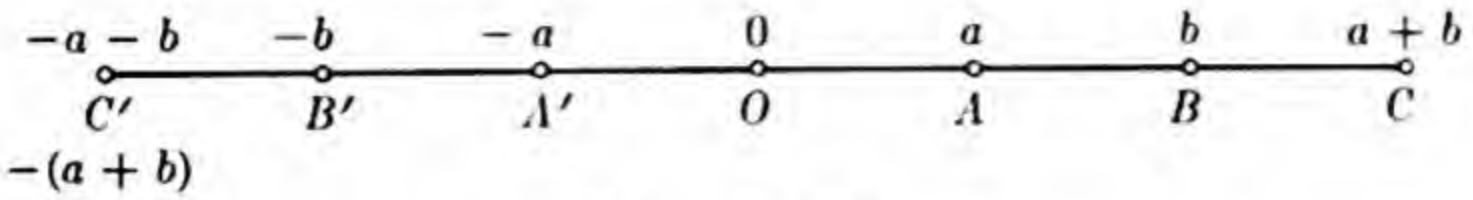


FIGURE 1-3

lowing rules* apply for any a, b, and c:

Rule 1. $-(a + b) = (-a) + (-b) = -a - b;$ (1-4)

Rule 2. $b + (-a) = b - a = -(a - b);$ (1-5)

Rule 3. $(a + b) + c = a + (b + c) = a + b + c;$ (1-6)

Rule 4. $a + (b - c) = a + b - c;$ (1-7)

Rule 5. $a - (b + c) = a - b - c;$ (1-8)

Rule 6. $a - (b - c) = a - b + c.$ (1-9)

Rule 1 may be interpreted in terms of the geometric operation of reflection. To add two numbers and then form the negative of their sum, as in Fig. 1-3 let a and b correspond to the points A and B, find the point C with coordinate $a + b$, then reflect this point on the origin to find C' with coordinate $-(a + b)$. Alternatively, first reflect A and B on the origin to find the points $A'(-a)$ and $B'(-b)$ and then find the point, again C', with coordinates $(-a) + (-b)$ or $-a - b$. For the problem of arithmetic in which -5 and -3 are added to give -8, the rule states that we first add 5 and 3 and then form the negative of the sum.

The first part of Rule 2 is a repetition of Corollary 1-1. The other part may be interpreted in terms of directed distance. Since $d(AB) = b - a$ and $d(BA) = a - b$, it follows from the rule that $d(AB) = -d(BA)$.

If letters are used to represent numbers, it is not known which is the larger; but for natural numbers, as in elementary arithmetic, one can determine which is larger. If $b > a$, the form $b + (-a) = b - a$ is used, whereas if $a > b$, then the form $b + (-a) = -(a - b)$ is used. For example, in elementary problems: (1) add 9 and -3 to get 6, subtract 3 from 9; (2) add 3 and -9 to get -6, subtract 3 from 9 and form the negative of this difference.

In arithmetic, Rules 1 and 2 are often stated in the following forms:

(1) *To add two negative numbers, add their absolute values and prefix the negative sign.*

---

* Rule 3 is restated here for convenience of reference; it is the Associative Law of Addition, A3.

(2) *To add a positive number and a negative number, find the difference of their absolute values and use the sign of the one with the larger absolute value.*

All of the Rules 1 through 6 can be stated as the rules for removal or insertion of parentheses:

Parentheses preceded by a + sign may be removed or inserted by retaining the sign of each term enclosed in the parentheses.

Parentheses preceded by a − sign may be removed or inserted by changing the sign of each term enclosed in the parentheses.

Since $a + b = b + a$, it is understood that when no sign appears, a + sign is implied.

The following convention for the use of signs of grouping is adopted:

If several signs of grouping appear, one within the others, the operations within the innermost enclosures are to be performed first.

For example,

$$\begin{aligned} x - \{d - [a - (b - c)]\} &= x - \{d - [a - b + c]\} \\ &= x - \{d - a + b - c\} \\ &= x - d + a - b + c. \end{aligned}$$

We first removed the parentheses, ( ); then the brackets, [ ]; and then the braces, { }. Parentheses and brackets can then be inserted:

$$\begin{aligned} x - d + a - b + c &= x + (a + c) - (b + d) \\ &= x - [(b + d) - (a + c)], \end{aligned}$$

to yield forms which are useful according as $(a + c) > (b + d)$ or $(b + d) > (a + c)$.

Problem Set 1–3

1. For the number scale given in Fig. 1–4, find the following directed distances and lengths:

$$d(AB),\ d(AC),\ d(CA),\ d(A'B'),\ d(BA),$$

$$AB,\ BA,\ A'B',\ AC,\ A'C.$$

2. For the number scale of Fig. 1–4, which of the following statements are true?

(a) $d(AB) + d(BC) = d(AC)$

(b) $AB + BC = AC$

(c) $d(CA) + d(AB) = d(CB)$

(d) $CA + AB = CB$

(e) $d(CA') + d(A'O) = d(OC)$

(f) $CA' + A'O = CO$

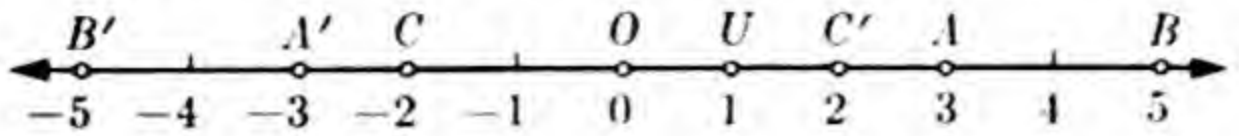


FIGURE 1–4

3. Write in symbols the algebraic equivalents of the following statements, using the coordinates given in problem 1.

(a) $X(x)$ is between B and C'

(b) $X(x)$ is between B and C

(c) the set of all points between B' and C'

(d) the segment $B'C'$; the segment AC

(e) the ray whose endpoint is A and which contains B

(f) the ray whose endpoint is C and which contains A'

4. Perform the following additions. Justify your results in terms of the Rules 1–6 or equivalent statements.

(a) $\begin{array}{r} -7 \\ -6 \\ \hline \end{array}$ (b) $\begin{array}{r} -7 \\ 4 \\ \hline \end{array}$ (c) $\begin{array}{r} 7 \\ -4 \\ \hline \end{array}$

(d) $\begin{array}{r} -9 \\ -5 \\ \hline \end{array}$ (e) $\begin{array}{r} -9 \\ 5 \\ \hline \end{array}$ (f) $\begin{array}{r} 9 \\ -5 \\ \hline \end{array}$

(g) $\begin{array}{r} -7 \\ -6 \\ 9 \\ \hline \end{array}$ (h) $\begin{array}{r} -7 \\ 6 \\ -9 \\ \hline \end{array}$ (i) $\begin{array}{r} 7 \\ -6 \\ -9 \\ \hline \end{array}$

(j) $\begin{array}{r} -12 \\ -6 \\ 18 \\ \hline \end{array}$ (k) $\begin{array}{r} -12 \\ 6 \\ -18 \\ \hline \end{array}$ (l) $\begin{array}{r} 12 \\ -6 \\ -18 \\ \hline \end{array}$

5. Perform the following subtractions. Justify your results in terms of the Rules 1–6 or equivalent statements.

(a) $\begin{array}{r} -7 \\ -6 \\ \hline \end{array}$ (b) $\begin{array}{r} -7 \\ -9 \\ \hline \end{array}$ (c) $\begin{array}{r} -7 \\ 4 \\ \hline \end{array}$ (d) $\begin{array}{r} 7 \\ 9 \\ \hline \end{array}$

(e) $\begin{array}{r} -12 \\ 10 \\ \hline \end{array}$ (f) $\begin{array}{r} 25 \\ -13 \\ \hline \end{array}$ (g) $\begin{array}{r} -6 \\ -13 \\ \hline \end{array}$ (h) $\begin{array}{r} -8 \\ -8 \\ \hline \end{array}$

6. In the following, remove all signs of aggregation and simplify results. Use the first part of each problem to check the second part.

(a) $7 - [5 - (4 + 8)]$, $a - [b - (c + d)]$

(b) $3 - [7 - (4 - 8)]$, $a - [b - (c - d)]$

(c) $8 - \{8 - [8 - (8 - 3)]\}$, $a - \{a - [a - (a - b)]\}$

(d) $6 - \{6 - [6 - (6 + 12)]\}$, $a - \{a - [a - (a + b)]\}$ (cont.)

(e) $5 - \{7 - [7 - (4 - 5)]\}$, $a - \{b - [b - (c - a)]\}$
(f) $6 - \{9 - [9 - (2 + 6)]\}$, $a - \{b - [b - (c + a)]\}$
(g) $x - \{8 - [4 - (3 - 7)]\}$, $x - \{d - [a - (b - c)]\}$
(h) $x - \{4 - [7 + (8 - 12)]\}$, $x - \{d - [a + (b - c)]\}$

1–9 Multiplication by an integer. In elementary arithmetic where only positive integers are involved, multiplication is treated as abbreviated addition in the sense that a number such as $7a$ is considered the result of the successive addition of 7 numbers all of which are equal to a. This concept for positive integers is now generalized.

DEFINITION. If a is any real number and n is a positive integer, then the product $n \times a = na$ is given by the equation

$$na = (a + a + a + \ldots \text{to } n \text{ terms}). \tag{1–10}$$

This definition would be meaningless unless n is a positive integer (a counting number), but it admits easy extension to give meaning to $-na$.

DEFINITION. If a is any real number and n is a positive integer, then $-na$ is the additive inverse of na, that is, $-na = (-(na))$ is such that

$$(-(na)) + na = 0. \tag{1–11}$$

The inner parentheses are used to indicate that the product na is formed first. The outer parentheses are parts of the notation for the additive inverse of a number. Since no misunderstanding can occur, and in agreement with the rules for omission of parentheses, both sets of parentheses may be omitted.

For such products, the following rules for the omission or insertion of parentheses are stated:

For any real number a and for any positive integers m and n,

$$ma + na = (m + n)a, \tag{1–12}$$

$$ma - na = (m - n)a. \tag{1–13}$$

Proof of Eq. (1–12):

$$\begin{aligned} ma + na &= (a + a + \cdots \text{to } m \text{ terms}) + (a + a \cdots \text{to } n \text{ terms}) \\ &= (a + a + a + \cdots \text{to } (m + n) \text{ terms}) \\ &= (m + n)a. \end{aligned}$$

Proof of Eq. (1–13):

If $m > n$, so that $m - n$ is a positive integer,

$$\begin{aligned} ma - na &= (a + a + \cdots \text{ to } m \text{ terms}) - (a + a + \cdots \text{ to } n \text{ terms}) \\ &= (a + a + a + \cdots \text{ to } (m - n) \text{ terms}) \\ &= (m - n)a. \end{aligned}$$

If $n > m$, so that $n - m$ is positive, Eq. (1–5) is used twice and then Eq. (1–12) is used:

$$ma - na = -(na - ma) = -(n - m)a = (m - n)a.$$

Successive applications of these laws permit addition and subtraction of algebraic expressions that involve letters, representing real numbers, which are multiplied by positive integers. Such operations may be shown in column form or the additions and subtractions may be indicated on a line through the use of parentheses.

EXAMPLE 1–1. Add: $a = b = c = 1$

$3a - 5b + 2c$	0
$-4a + b - 3c$	-6
$5a + 2b + c$	8
$4a - 2b$	2 Check.

EXAMPLE 1–2. Subtract: $a = 3, b = 2, c = 1$

$3a - 5b + 2c$	1
$-4a + b - 3c$	-13
$7a - 6b + 5c$	14 Check.
$21 - 12 + 5 =$	14

For the first column, we have $3a - (-4a) = 3a + 4a$ because of Eq. (1–1); for the second column, we have $-5b - b = -(5b + b)$, because of Eq. (1–4); for the third column, we have $2c - (-3c) = 2c + 3c$. The final answer is obtained by using Eq. (1–12).

EXAMPLE 1–3. Remove all parentheses and simplify results:

$$x = (3a - 5b + 2c) - (4a - b + 3c) + (5a - 2b + 3c)$$

Solution:
$$\begin{aligned} x &= 3a - 5b + 2c - 4a + b - 3c + 5a - 2b + 3c \\ &= (3 - 4 + 5)a + (-5 + 1 - 2)b + (2 - 3 + 3)c \\ &= 4a - 6b + 2c. \end{aligned}$$

Since the rules used are valid for all real values of a, b, c, the operations hold if special values are assigned to a, b, and c. The rule of substitution gives the same result whether we compute first and then add (or subtract), or add first and then compute the final value for these special values. Such checks are given for Examples 1–1 and 1–2 above. Even if the two results check, there is a possibility that we have made compensating errors; but if the results do not check, we are sure that an error has been made.

Problem Set 1–4

1. Add:

(a)
$$\begin{array}{r} 6a - 5b + 2c \\ -5a + 3b - 4c \\ a - 5b - 6c \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 2a - 4b + 3c \\ -4a + 6b - 2c \\ a - b - c \\ \hline \end{array}$$

(c)
$$\begin{array}{r} -3x + 4y - 2z \\ 4x - 2y - 3z \\ -2x - 3y + 4z \\ \hline \end{array}$$

(d)
$$\begin{array}{r} 7x - 9y + 2z \\ 5x + 3y - 7z \\ -10x - 6y - 6z \\ \hline \end{array}$$

(e)
$$\begin{array}{r} 3p + 7q + 4r \\ -5p - 9q + 7r \\ p - 2q - 8r \\ \hline \end{array}$$

(f)
$$\begin{array}{r} 3p - 7q - 4r \\ 5p + 9q + 7r \\ -8p - 2q - 3r \\ \hline \end{array}$$

2. Add (x^2, xy, etc. are to be considered as compound symbols for numbers):

(a)
$$\begin{array}{r} 4x^2 - 6x + 5 \\ -5x^2 - 6x + 4 \\ 4x^2 - 6x - 3 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} -4x^2 - 6x + 5 \\ 7x^2 - 2x + 3 \\ 3x^2 + 5x - 9 \\ \hline \end{array}$$

(c)
$$\begin{array}{r} 4x^2 - 6xy + 5y^2 \\ -5x^2 - 6xy + 4y^2 \\ 4x^2 - 6xy - 3y^2 \\ \hline \end{array}$$

(d)
$$\begin{array}{r} -4x^2 - 6xy + 5y^2 \\ -7x^2 + 2xy - 3y^2 \\ 5x^2 + 7xy - 8y^2 \\ \hline \end{array}$$

3. Subtract:

(a)
$$\begin{array}{r} 6a - 5b + 2c \\ -5a + 3b + 4c \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 2a - 4b + 3c \\ -4a + 6b - 2c \\ \hline \end{array}$$

(c)
$$\begin{array}{r} -6x - 4y - 3z \\ -2x + 4y - 3z \\ \hline \end{array}$$

(d)
$$\begin{array}{r} 8x + 9y - 7z \\ 9x - 7y + 8z \\ \hline \end{array}$$

(e)
$$\begin{array}{r} -5p - 9q + 7r \\ p - 2q - 8r \\ \hline \end{array}$$

(f)
$$\begin{array}{r} 3p - 7q - 4r \\ 5p + 9q + 7r \\ \hline \end{array}$$

(g)
$$\begin{array}{r} -5x^2 - 6x + 4 \\ 4x^2 - 6x - 3 \\ \hline \end{array}$$

(h)
$$\begin{array}{r} -4x^2 - 6x + 5 \\ 7x^2 - 2x + 3 \\ \hline \end{array}$$

(i)
$$\begin{array}{r} 4x^2 - 6xy + 5y^2 \\ -5x^2 - 6xy + 4y^2 \\ \hline \end{array}$$

(j)
$$\begin{array}{r} -4x^2 - 6xy + 5y^2 \\ -7x^2 - 2xy + 8y^2 \\ \hline \end{array}$$

4. In the following, remove all signs of aggregation and simplify results. Check each problem, using $a = 2$, $b = 3$, $c = 4$, or $x = 2$, $y = 3$, $z = 4$.

(a) $[3a - (5b - 2c)] - [4a - (b - 3c)] + [5a - (2b - 3c)]$

(b) $[3a - (5b - 2c)] - [4a - (b + 3c)] + [5a - (2b + 3c)]$

(c) $(-3a - 5b + 2c) - (4a + b - 3c) + (5a + 3b - 2c)$

(d) $(-3a - [5b + 2c]) - (4a + [b - 3c]) + (5a - [3b - 2c])$

(e) $[2x - (3y + 4z)] - [3y - (4x - 4z)]$

(f) $\{(2x - 3y) - (3y + 4z)\} - \{2x - (3x - 4y - 2z)\}$

CHAPTER 2

MULTIPLICATION

2–1 Multiplication of real numbers. There are many similarities and analogies between the laws of multiplication and those of addition. Some of the laws can be proved on the basis of the more fundamental ones, and such proofs are presented in Chapter 7. Other laws, concerning special numbers, will be proved here when their importance and simplicity indicate the desirability for doing so.

M1. CLOSURE.

If a and b are real numbers, then there exists a real number called their product. This is represented by $a \times b$, $a \cdot b$, or by placing the letters in juxtaposition as ab.

M2. COMMUTATIVE.

Multiplication is commutative. That is, for any real numbers a and b,

$$ab = ba.$$

M3. ASSOCIATIVE.

Multiplication is associative. That is, for any real numbers a, b, c,

$$(ab)c = a(bc) = abc.$$

Since the order of grouping is immaterial, such a product is written without parentheses.

When laws M2 and M3 are combined and extended to more than three numbers, the order in which the multiplication is done is immaterial. For example:

$$abcd = [(ab)c]d = [a(bc)]d = (ab)(cd) = (bd)(ca) = [(bd)a]c = (da)(cb).$$

M4. ONE.

There exists a unique real number, called one and represented by **1**, *such that for any real number a,*

$$a \cdot 1 = 1 \cdot a = a.$$

M5. MULTIPLICATIVE INVERSE.

For any real number a different from zero, there exists a unique number, called the multiplicative inverse of a or the reciprocal of a

and represented by the symbol $(1/a)$, *such that*

$$a\,(1/a) = 1,\ (a \neq 0).$$

In other words, the equation $ax = 1$, $(a \neq 0)$, has a unique solution which is represented by the symbol

$$(1/a) \text{ or } \frac{1}{a}.$$

This definition and law will be discussed and exploited in Chapter 3, "Division and Fractions."

M6. MULTIPLICATION BY A NUMBER.

For any real numbers a, b, c, *if* $a = b$, *then* $ac = bc$, *and conversely.*

M7. LAW OF CANCELLATION FOR MULTIPLICATION.

For any real numbers a, b, c, *such that* $c \neq 0$, *if* $ac = bc$, *then* $a = b$.

Laws M6, M7, and their extensions are often stated in the form:

If equals be multiplied or divided by equals, the results are equal.

These laws are assumed valid and used here to develop skill in solving equations or sets of equations. In Chapter 7 it is shown that they can be proved on the basis of the other postulates of multiplication and the Principle of Substitution.

This section is concluded with the statement that multiplication is distributive with respect to addition. It is to be emphasized that having assumed certain laws concerning the operation of addition and having assumed certain laws concerning the operation of multiplication, it is necessary to have at least one assumption which relates these two operations.

M8. DISTRIBUTIVE.

For any real numbers a, b, c,

$$(a + b)c = c(a + b) = ac + bc.$$

This law is the basic idea behind multiplication of numbers of two or more digits, the use of parentheses, factoring, multiplication of polynomials, and the manipulation of fractions.

2–2 Multiplication by zero. The numbers zero, 0, and one, 1, are so special and so important in algebra that some of their important properties are proved. The number zero was defined through use of the equation

$a + 0 = a$. Likewise, the additive inverse of a was defined by the equation $(-a) + a = 0$, so that the definition of subtraction implies that $(-a) = 0 - a$.

Two important theorems which involve 0 and the operation of multiplication are now proved.

THEOREM 2–1. *For any* a, $a \cdot 0 = 0$.

Proof.

$1 + 0 = 1$	Definition of zero;
$a(1 + 0) = a \cdot 1$	Multiplication by a (M6);
$\cdot a \cdot 1 + a \cdot 0 = a \cdot 1$	Distributive law (M8);
$= a \cdot 1 + 0$	Definition of zero.

Therefore $a \cdot 0 = 0$ Law of cancellation for addition.

COROLLARY. $0 \cdot a = 0$.

THEOREM 2–2. *If* $ab = 0$, *then either* $a = 0$ *or* $b = 0$.

Proof. If $ab = 0$ and $a = 0$, there is nothing further to prove. Suppose then that* $a \neq 0$ and

$ab = 0$	Hypothesis;
$a \cdot 0 = 0$	Theorem 2–1;
$ab = a \cdot 0$	Transitive law (E3).

Therefore $b = 0$ Law of cancellation (M7)

and the theorem is established.

Theorems 2–1 and 2–2 may be combined to read: $ab = 0$ *if and only if* $a = 0$ or $b = 0$.

Nothing in the foregoing proof implies that a and b might not both be zero. To solve the equation $x(x - 4) = 0$, $x = 0$ would be one solution, and if $x \neq 0$, then we must have $x - 4 = 0$, or $x = 4$ is another solution. It is not difficult to show that these are the only solutions. For if $y \neq 0$ or $y \neq 4$ were a solution, then $y(y - 4) = 0$, and Theorem 2–2 shows that either $y = 0$ or $y = 4$, which is impossible. The method of proof by contradiction was used in the preceding sentence and another proof will now be given which uses this method. We prove that the numbers 0 and 1 are different. "Zero" was defined by means of the operation of addition, and "one" was defined by means of the operation of multiplication, so there is no *a priori* reason why they might not be the same number.

* The symbol $\neq$ is read: "not equal."

THEOREM 2–3. $0 \neq 1$.

Proof. If $0 = 1$, then

$$a \cdot 0 = a \cdot 1, \quad \text{for any } a \neq 0 \quad \text{(Multiplication by } a\text{, M6)}.$$

But

$$a \cdot 0 = 0 \quad \text{(Theorem 2–1)}$$

and

$$a \cdot 1 = a \quad \text{(Law of one, M4)}.$$

Hence

$$0 = a,$$

which is impossible, and hence $0 \neq 1$.

2–3 Rules of sign. The laws of multiplication do not specify whether the numbers considered are positive or negative, but any real numbers a and b have additive inverses $(-a)$ and $(-b)$, and it is necessary to know how to combine such numbers under multiplication. First consider the additive inverse of 1, namely, (-1). Two interpretations of the symbol $-a$ have been given: as the additive inverse of a, and as a part of the operation of subtracting a. A third interpretation, involving multiplication, is now given.

THEOREM 2–4. $(-1)a = -a$, *for any* a.

Proof. Start with the expression $a + (-1)a$:

$$\begin{aligned} a + (-1)a &= 1 \cdot a + (-1)a && \text{Law of one and principle of substitution;} \\ &= [1 + (-1)]a && \text{Distributive law;} \\ &= 0 \cdot a = 0 && \text{Additive inverse and Theorem 2–1.} \end{aligned}$$

Apply the definition of subtraction to

$$a + (-1)a = 0.$$

Then

$$(-1)a = -a. \tag{2–1}$$

THEOREM 2–5. *If* a *and* b *are real numbers, then*

$$a(-b) = -(ab), \tag{2–2}$$

$$(-a)(-b) = ab. \tag{2–3}$$

Proof. $a(-b) = a(-1)b = (-1)(ab) = -(ab).$

Also $(-a)(-b) = (-1)a(-b) = (-1)(-ab) = -(-ab) = ab.$

COROLLARIES. $(-1)(1) = -1$; $(-1)(-1) = 1$. (2–4)

M9. PRODUCT OF POSITIVES.

The product of two positive numbers is positive.

If in Eqs. (2–2) and (2–3) a and b are positive numbers, the above rules cover the special cases where the product of a negative number and a positive number is negative, and the product of two negative numbers is positive.

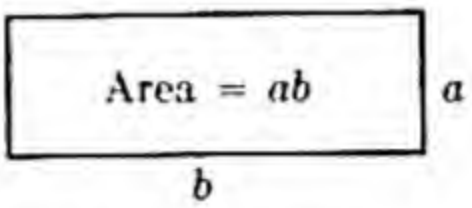


FIGURE 2–1

A geometric representation of the product of two *positive* numbers is a rectangle with the area of the rectangle equal to a positive number which is the product of the *lengths* of two adjacent sides. It is postulated that the area of a rectangle is the sum of areas of rectangles into which it is subdivided. The validity of Eq. (2–2) is seen as follows: Take a square with one side of length a and let b be a positive number less than a. Divide the square into two rectangles (Fig. 2–2) with bases of length b and $a - b$. In terms of area:

$$\begin{aligned} a^2 &= ab + a(a - b) \\ &= ab + a[a + (-b)] && \text{See Eq. (1–3)} \\ &= ab + a^2 + a(-b) && \text{Distributive law} \\ 0 &= ab + a(-b) && \text{Cancellation} \end{aligned}$$

But $0 = ab + (-ab)$ Additive inverse

and it follows that from the uniqueness of the additive inverse that

$$a(-b) = -(ab) = -ab. \tag{2–2}$$

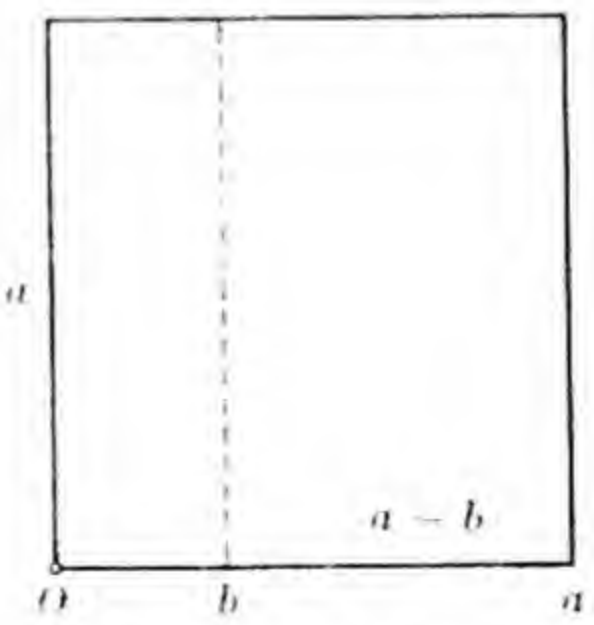


FIGURE 2–2

2–4 Removal of parentheses. The rules for removal of signs of grouping are based on the distributive law of multiplication. Thus:

$$+(a+b) = +1 \cdot (a+b) = a+b,$$
$$-(a+b) = (-1)(a+b) = (-1)a + (-1)b = -a-b.$$

Similarly

$$-(a-b) = (-1)(a-b) = (-1)[a+(-b)]$$
$$= (-1)a + (-1)(-b) = -a+b.$$

In general, in order to remove or insert signs of grouping, the distributive law of multiplication and Theorem 2–5 are used.

EXAMPLE 2–1. Simplify $x = a + 2[b - (c - a + 3b)]$.

Solution.
$$x = a + 2[b + (-1)(c + (-a) + 3b)]$$
$$= a + 2[b - c + a - 3b]$$
$$= a + 2b - 2c + 2a - 6b$$
$$= 3a - 4b + 2c.$$

Check. Let $a = 4$, $b = 3$, $c = 2$. The original form gives

$$x = 4 + 2[3 - (2 - 4 + 9)] = 4 + 2[-4] = -4;$$

the final form gives $x = 12 - 12 - 4 = -4$. Check.

PROBLEM SET 2–1

1. Simplify the following expressions by removing all signs of grouping.

(a) $7x - 8 + 2(3 - 4x)$
(b) $-(2x - 4) + (3 - 4x)$
(c) $-(2a - 3b) + (4a - 2b)$
(d) $3(a - b) - 2(a + b)$
(e) $-(2x + 4) - 2(3x - 5) + 3(x - 6)$
(f) $2(4 - 2x) - 3(3 - 3x) + 2(2x - 6)$
(g) $-(a + b - c) - (a - b + c) - (-a + b + c)$
(h) $2(a - 2b + c) + 3(2a - b + c) - 4(-2a + b - c)$
(i) $x = 3a - 4(3b - a) + 2(a - 2b)$
(j) $y = 4x - 2(x - 2) - 3(3 - 2x)$

After solving problems (k), (l), (m), (n), check each problem using $a = 4$, $b = 3$, $c = 2$.

(k) $x = -2a + 3[b - (2c - 3a + b)]$

(l) $x = -(2a - 3b) + 3[(a - 2b) - (2c + 3a - b)]$

(m) $y = c + 2\{(b - a) - [3b - (c - a)]\}$

(n) $y = (a + b - c) + [2(a - b + c) - 3(-a + b + c)]$

2. Which of the following statements are true and which are false?

(a) $3(4 + 5) = 3 \cdot 4 + 3 \cdot 5$

(b) $3(4 + 5) = 3 \cdot 4 + 5$

(c) $3 + (4 \cdot 5) = (3 + 4) + 3 \cdot 5$

(d) $3 + (4 \cdot 5) = 3 \cdot 4 + 3 \cdot 5$

(e) For any a, b, c, $a + (b \cdot c) = (a + b)c$. Is this equation ever true?

(f) For any a, b, c, $a + (b \cdot c) = a \cdot b + a \cdot c$. Is this equation ever true? Give an example.

(g) For any a, b, c, $a + (b \cdot c) = (a + b)(a + c)$. Is addition distributive with respect to multiplication? Explain. Is the equation given above ever true?

3. By means of the Associative Law of Multiplication, prove

$$(ab)(cd) = [(ab)c]d = [a(bc)]d.$$

4. (a) Prove that (Corollary to Theorem 2-5): $(-1)(-1) = 1$.

(b) It has been assumed that 1 is a positive number. Use the above equation to prove that 1 is not a negative number and hence must be a positive number. Cite any theorems or assumptions that are used. (Suggestion: M9 and Theorem 2-3.)

5. Prove Theorem 2-4: $(-1)a = -a$ by starting with $(-1) + 1 = 0$ and multiplying by a.

2-5 Positive integral exponents. The product of n numbers all of which are equal is indicated by using the positive integer n as an *exponent*. Thus:

$$a \cdot a = a^2, \quad a \cdot a \cdot a = a^3, \quad a \cdot a \ldots \text{to } n \text{ factors is } a^n,$$

where a^2, a^3, a^n are read "a squared," "a cubed," "a to the *power* n."

If n and m are positive integers, then for any a and b,

$$a^n \cdot a^m = a^{n+m}; \tag{2-5}$$

$$(a^n)^m = (a^n)(a^n) \cdots \text{to } m \text{ factors} = a^{nm}; \tag{2-6}$$

$$(ab)^n = a^n b^m. \tag{2-7}$$

Equations (2–5) and (2–6) follow directly from the above definition of exponents. Equation (2–7) is a consequence of this definition and the laws of multiplication.

$$\begin{aligned}(ab)^n &= (ab)(ab)\cdots \text{to } n \text{ factors}\\ &= (a\cdot a\cdots \text{to } n \text{ factors})\ (b\cdot b\cdots \text{to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

An algebraic expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n,$$

where n is a positive integer and $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n$ are constants such that $a_0 \neq 0$, is called a *polynomial* in the variable x of *degree* n.

The polynomial consists of several terms connected by signs of addition or subtraction. A polynomial of one term is called a *monomial;* a polynomial of two terms is called a *binomial*, and so on.

An algebraic expression of the form cx^ny^m, where c is a constant, x and y are variables, and n and m are positive integers, is called a *monomial in* x *and* y; a sum of such terms with different values for n and m is called a *polynomial in* x *and* y. The sum of the exponents, $n + m$, is called the degree of the term. If all terms have the same degree, the polynomial is said to be *homogeneous of degree* $n + m$. Thus

$3x^4$, $-5x^2y^2$, xy^4 are monomials;

$3x^4 - 5x^3$; $x^2 + 2y$ are binomials;

$3x^5 - 2x^4 + 2x + 1$ is a polynomial in x of degree 5;

$x^4 - 5x^2y^2 + 6y^4$ is a homogeneous polynomial of degree 4 in the variables x and y.

2–6 Multiplication of polynomials. Special products. To multiply two polynomials, multiply one polynomial by each term of the second polynomial and add the results.

EXAMPLE 2–2. Multiply $(x^2 - 2x + 3)(x^2 - x + 2)$.

$$\begin{array}{r} x^2 - 2x + 3 \\ x^2 - x + 2 \\ \hline x^4 - 2x^3 + 3x^2 \\ \quad - x^3 + 2x^2 - 3x \\ \quad\quad + 2x^2 - 4x + 6 \\ \hline x^4 - 3x^3 + 7x^2 - 7x + 6 \end{array}$$ Ans.

As a check we could use $x = 1$. $(1 - 2 + 3)(1 - 1 + 2) = 4$.

$$1 - 3 + 7 - 7 + 6 = 4. \quad \text{Check.}$$

EXAMPLE 2–3. Expand $(a - b)^4 = [(a - b)^2]^2$.

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ \quad - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array} \qquad \begin{array}{r} a^2 - 2ab + b^2 \\ a^2 - 2ab + b^2 \\ \hline a^4 - 2a^3b + a^2b^2 \\ \quad - 2a^3b + 4a^2b^2 - 2ab^3 \\ \quad + a^2b^2 - 2ab^3 + b^4 \\ \hline a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \end{array}$$

As a check we could use $a = 3$, $b = 1$. $(3 - 1)^4 = 16$.

$$81 - 108 + 54 - 12 + 1 = 136 - 120 = 16. \quad \text{Check.}$$

Certain special products occur so frequently that familiarity with these products saves time and makes for accuracy in algebraic manipulation. Each such product may be verified by performing the indicated multiplications.

For any a, b, c, d, x,

$$(a + b)^2 = a^2 + 2ab + b^2, \tag{2–8}$$

$$(a - b)^2 = a^2 - 2ab + b^2, \tag{2–9}$$

$$(a + b)(a - b) = a^2 - b^2, \tag{2–10}$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab, \tag{2–11}$$

$$(cx + a)(dx + b) = cdx^2 + (ad + bc)x + ab, \tag{2–12}$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \tag{2–13}$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3, \tag{2–14}$$

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3, \tag{2–15}$$

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3. \tag{2–16}$$

EXAMPLE 2–4.

$$\begin{aligned} (2x^2 - 3y)^3 &= (2x^2)^3 - 3(2x^2)^2(3y) + 3(2x^2)(3y)^2 - (3y)^3 \\ &= 8x^6 - 36x^4y + 54x^2y^2 - 27y^3. \end{aligned}$$

Problem Set 2-2

1. Simplify the following products.
   (a) $(-2x^2y^3)(5xy^2)$
   (b) $(-2x^2y^3)^2(5x^2y)$
   (c) $(7xy)^2(5x^2y)^3$
   (d) $(-2x^2y)(-3xy^2)(6xy)^2$

2. Perform the indicated multiplications.
   (a) $(x^3 + 5x^2 - 2x - 5)(3x + 4)$
   (b) $(x^2 - 7x + 11)(5 - 3x)$
   (c) $(x + 3)(x - 2)(x + 2)$; check, using $x = 1$
   (d) $(x^2 - 2x + 3)(2x^2 - x - 4)$
   (e) $(a^2 + a + 1)(a^2 - a + 1)$; check, using $a = 2$
   (f) $(y^2 + y - 1)(y^2 - y + 1)$; check, using $y = 3$

3. Perform the indicated multiplications.
   (a) $(2x - 3y)(5x + 4y)$
   (b) $(3a - 5b)(3a + 5b)$
   (c) $(a^3 + a^2b + ab^2 + b^3)(a - b)$
   (d) $(a^3 - a^2b + ab^2 - b^3)(a^2 - b^2)$
   (e) $(2x^3 + 3xy + y^2)(x - y^2)$

4. Use the special product formulas to find the following products.
   (a) $(3x - 2y^2)^2$
   (b) $(3x - 2y^2)(3x + 2y^2)$
   (c) $(2x + 3y)^3$
   (d) $(2x + 3y)(2x + 5y)$
   (e) $(3x + 5)(x + a)$ for any x and the special values: $a = 3$, $a = 2$, $a = -2$, $a = -3$

5. Expand $(a + b + c)^2$ and use the result to find $(3x - 2y + 4z)^2$.

6. Find the following products.
   (a) $(x - 1)(x - 2)(x - 3)$
   (b) $(x + 1)(x + 2)(x + 3)$
   (c) $(x - 1)(x + 1)(x - 2)(x + 2)$

7. Verify the special products given by Eqs. (2-15) and (2-16). Check these formulas using $a = b$ and also using $a = -b$.

8. Prove $(a + b)(x + y) = ax + ay + bx + by$. Justify each step in the process.

CHAPTER 3

DIVISION AND FRACTIONS

3–1 Reciprocal of a number. Law M5 was stated as follows:

M5. *For any real number* b, $b \neq 0$, *there exists a unique number called the multiplicative inverse of* b *(or the reciprocal of* b*)—denoted by the symbol* $(1/b)$*—such that*

$$b(1/b) = 1, \qquad (b \neq 0).$$

THEOREM 3–1.

$$\frac{1}{(1/b)} = b, \qquad (b \neq 0).$$

The reciprocal of the reciprocal of a number is the number.

Proof. $\dfrac{1}{(1/b)}$ is a number which when multiplied by $(1/b)$ gives 1, that is,

$$(1/b)\,\frac{1}{(1/b)} = 1.$$

But

$$(1/b)b = 1 \qquad \text{(M5)}.$$

Since $(1/b)$ has a unique reciprocal,

$$\frac{1}{(1/b)} = b.$$

THEOREM 3–2.

$$\left(\frac{1}{a}\right)\left(\frac{1}{b}\right) = \frac{1}{(ab)}, \quad (ab \neq 0).$$

The product of the reciprocals of two numbers is the reciprocal of the product of the numbers.

Proof. Multiply the product on the left by ab and then apply the commutative and associative laws of multiplication.

$$(ab)\left(\frac{1}{a}\right)\left(\frac{1}{b}\right) = a\left(\frac{1}{a}\right)b\left(\frac{1}{b}\right) = 1 \cdot 1 = 1,$$

$$(ab)\,\frac{1}{(ab)} = 1 \quad \text{(M5)}.$$

Hence $$\left(\frac{1}{a}\right)\left(\frac{1}{b}\right) = \frac{1}{(ab)}.$$

3-2 Division as the inverse of multiplication. If a and b are real numbers and $b \neq 0$, the result of dividing a by b is a number c such that

$$a\left(\frac{1}{b}\right) = c.$$

This division is represented by such symbols as

$$a \div b, \qquad \frac{a}{b}, \qquad a/b.$$

It is called the *quotient* of a by b, the *ratio of* a to b, or the *fraction* a over b.

THEOREM 3-3. *$a/b = c$, $(b \neq 0)$, if and only if $a = bc$.*

Proof. If $a/b = c$, multiply both members of this equation by b and simplify:

$$b\left(\frac{a}{b}\right) = b\left(\frac{1}{b}\right)a = bc$$

$$1 \cdot a = a = bc.$$

If $a = bc$, multiply both members of this equation by $(1/b)$ and simplify:

$$\left(\frac{1}{b}\right)a = \frac{a}{b} = \left(\frac{1}{b}\right)bc = 1 \cdot c = c, \quad \text{or} \quad \frac{a}{b} = c.$$

COROLLARY 1. $\frac{a}{a} = 1; \quad \frac{a}{1} = a.$

Both of these results follow from the equation $a = 1 \cdot a$.

COROLLARY 2. $\frac{1}{(-1)} = -1; \quad \frac{(-1)}{(-1)} = 1.$

These results follow from the equations

$$1 = (-1)(-1) \quad \text{and} \quad (-1) = (-1) \cdot 1 \qquad \text{(Theorem 2-5).}$$

COROLLARY 3. $0/b = 0, \quad b \neq 0.$

Proof. $0/b = 0(1/b) = 0$ (Theorem 2-1).

It is noted that division by zero is not defined, since the reciprocal of zero was not defined. Although such a symbol as $a/0$ might be written, it is meaningless.

3–3 Rules of sign. For any real numbers a and b, $(b \neq 0)$,

$$\frac{(-a)}{b} = \frac{a}{(-b)} = -\left(\frac{a}{b}\right); \quad \frac{(-a)}{(-b)} = \frac{a}{b}.$$

That $(-a)/b = -(a/b)$ follows from the fact that $(-c) = (-1)c$ (Theorem 2–4), the properties of reciprocals discussed in Section 3–1, and the fundamental laws of multiplication.

To prove $a/(-b) = -(a/b)$, note that

$$\frac{a}{(-b)} = \frac{a}{(-1)b} = a\frac{1}{(-1)b}$$
$$= a\left(\frac{1}{-1}\right)\left(\frac{1}{b}\right) = (-1)\left(\frac{a}{b}\right) = -\left(\frac{a}{b}\right).$$

To prove $(-a)/(-b) = a/b$, replace a by $-a$ in the foregoing to get

$$\frac{(-a)}{(-b)} = -\frac{(-a)}{b}$$
$$= -(-a)\frac{1}{b} = a\left(\frac{1}{b}\right) = \frac{a}{b}.$$

COROLLARY. *If a and b are positive numbers, the quotient $a/b = c$ is a positive number. The quotient of a positive and a negative number is negative, and the quotient of two negative numbers is positive.*

3–4 Equivalent fractions (change of form). A given fraction is equal to the fraction obtained by multiplying both its numerator and denominator by the same nonzero number.

$$\frac{a}{b} = \frac{ac}{bc}, \quad bc \neq 0. \tag{3–1}$$

Proof.

$$\frac{ac}{bc} = (ac)\frac{1}{(bc)} \qquad \text{Definition of division}$$
$$= ac\left(\frac{1}{b}\right)\left(\frac{1}{c}\right) \qquad \text{Theorem 3-2}$$
$$= a\left(\frac{1}{b}\right)c\left(\frac{1}{c}\right) \qquad \text{Commutative law}$$
$$= \frac{a}{b} \qquad \text{Definition of division, used twice.}$$

Since division by d $(d \neq 0)$ means multiplication by the reciprocal $(1/d)$, it also follows that

$$\frac{a}{b} = \frac{a/d}{b/d}, \quad bd \neq 0. \tag{3–2}$$

When fractions occur in problems of measurement, Eq. (3–1) represents a change of the unit of measure. Consider the following problem: How many square rubber tiles, $\frac{3}{4}$ ft on a side, are needed to cover a strip of flooring $10\frac{1}{2}$ ft long and $\frac{3}{4}$ ft wide?

If $a = 10\frac{1}{2}$ (ft) and $b = \frac{3}{4}$ (ft), then

$$\text{number of tiles} = \frac{a}{b} = \frac{10\frac{1}{2}}{\frac{3}{4}} = \frac{10\frac{1}{2}}{\frac{3}{4}} \cdot \frac{12}{12} = \frac{126}{9} = 14.$$

This is equivalent to expressing both lengths in inches:

$$A = 10\tfrac{1}{2} \cdot 12 = 126 \text{ (in.)}$$

$$B = \tfrac{3}{4} \cdot 12 = 9 \text{ (in.)}$$

$$\text{number of tiles} = \frac{126}{9} = 14.$$

The original problem could have been solved by multiplying both numerator and denominator by 4 instead of 12:

$$\text{number of tiles} = \frac{10\frac{1}{2}}{\frac{3}{4}} = \frac{21/2 \cdot 4}{\frac{3}{4} \cdot 4} = \frac{42}{3} = 14.$$

A reversal of the change of units from inches to feet would illustrate Eq. (3–2). If the strip is 126 in. long, then

$$\text{number of tiles} = \frac{126}{9} = \frac{126/12}{9/12} = \frac{21/2}{3/4} = \frac{21}{2} \cdot \frac{4}{3} = 14,$$

where a second application of Eq. (3–2) is used to simplify the fraction.

If Eq. (3–2) is applied to the quotient of two positive integral powers of a, and the fundamental definitions and properties of Section 2–5 are used, we have

$$\frac{a^n}{a^m} = \begin{cases} a^{n-m}, & \text{if } n > m \\ 1, & \text{if } n = m \\ 1/a^{m-n}, & \text{if } n < m, \end{cases} \tag{3–3}$$

where m and n are positive integers.

Similarly,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad (n \text{ a positive integer}, b \neq 0), \tag{3–4}$$

since

$$\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\cdots \text{ to } n \text{ factors} = a\left(\frac{1}{b}\right)a\left(\frac{1}{b}\right)\cdots$$

$$= (a\cdot a \ldots \text{ to } n \text{ factors})\left(\frac{1}{b}\cdot\frac{1}{b}\ldots \text{ to } n \text{ factors}\right) = \frac{a^n}{b^n}.$$

Problem Set 3–1

1. What are the reciprocals of 3, 5, 1/3, 3/5?

2. Prove (a) $\dfrac{(-a)}{b} = -\left(\dfrac{a}{b}\right)$; (b) $-\left(\dfrac{-a}{b}\right) = \dfrac{a}{b}$.

In the proofs explain what interpretations are given to the — sign.

3. Explain, on the basis of the definition $(1/b)b = 1$, why the reciprocal of a positive number is positive and the reciprocal of a negative number is negative.

4. Change the following fractions to forms where the numerator and denominator have no common factors and — signs appear only before the fractions.

(a) $\dfrac{-35ab^3}{21a^3b^2} = -\dfrac{5b}{3a^2}$ Why?

(b) $\dfrac{21a^2b}{-35ab}$ (c) $\dfrac{-15x^3}{-3x}$ (d) $\dfrac{-15x^3y}{-3x^4y^2}$

(e) $\dfrac{-36(-x)^3(-y)^2}{-15x^2y^3}$ (f) $\dfrac{-17x(-y)^3}{-29x^2y^2}$ (g) $-\dfrac{-12a^3b^2}{18a^4b^3}$

3–5 Algebra of fractions. The fundamental laws of multiplication, division, and addition of fractions may be written:

$$\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right) = \frac{(ac)}{(bd)}, \tag{3–5}$$

$$\left(\frac{a}{b}\right)\div\left(\frac{c}{d}\right) = \frac{(ad)}{(bc)}, \tag{3–6}$$

$$\left(\frac{a}{b}\right)+\left(\frac{c}{d}\right) = \frac{(ad+bc)}{(bd)}. \tag{3–7}$$

The parentheses could be omitted but are inserted here to indicate the order in which the operations are performed. Hereafter, the parentheses will be omitted except where it is important to indicate the order in which the operations are to be performed.

Equation (3–5) is an immediate consequence of the definition of division and the laws of multiplication:

$$\frac{a}{b} \times \frac{c}{d} = a \times \frac{1}{b} \times c \times \frac{1}{d} = ac \times \frac{1}{bd} = \frac{ac}{bd}.$$

Two proofs are given for Eq. (3–6). The first proof depends upon the definition of division as multiplication by the reciprocal:

$$\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \left(\frac{a}{b}\right) \cdot \frac{1}{(c/d)} = \left(\frac{a}{b}\right) \cdot \frac{1}{c(1/d)}$$

$$= \frac{a}{b} \cdot \frac{1 \cdot d}{c(1/d)d} = \left(\frac{a}{b}\right) \cdot \left(\frac{d}{c}\right)$$

$$= \frac{ad}{bc},$$

where the final step is justified by Eq. (3–5).

The second proof uses Theorem 3–3 and the law of multiplication by a constant, M6.

Let

$$\frac{a}{b} \div \frac{c}{d} = x.$$

Then

$$\frac{a}{b} = x \cdot \frac{c}{d} \qquad \text{Theorem 3–3,}$$

$$\frac{a}{b} \cdot d = x \cdot \frac{c}{d} \cdot d = xc \qquad \text{Multiplication by } d,$$

$$\frac{a}{b} \cdot d\left(\frac{1}{c}\right) = x \cdot c\left(\frac{1}{c}\right) \qquad \text{Multiplication by } \frac{1}{c},$$

$$\frac{a}{b} \cdot \frac{d}{c} = x = \frac{ad}{bc} \qquad \text{Definition of reciprocal and Eq. (3–5).}$$

Therefore

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}. \tag{3–6}$$

Equation (3–6) is often expressed in words as follows: *To divide one fraction by a second fraction, invert the second fraction and multiply.*

Change of form for the fractions and the distributive law of multiplication are used to prove Eq. (3–7).

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = ad\left(\frac{1}{bd}\right) + bc\left(\frac{1}{bd}\right)$$

$$= \frac{1}{bd}(ad + bc) = \frac{ad + bc}{bd}. \tag{3–7}$$

Equation (3–7) is expressed in words as follows: *To add two fractions, express each fraction as an equivalent fraction with a common denominator and add numerators.*

Subtraction of one fraction from another is accomplished by a change of sign followed by an addition.

Sections 3–1 to 3–5 apply when the numbers are fractions or algebraic expressions involving one or more variables. Fractions with zero denominator are excluded, and it is always understood that an algebraic fraction is not defined if the denominator is zero. Thus the fraction $(x^2 + 1)/(x - 1)$ is not defined if $x = 1$, and the fraction $(x^2 - y^2)/xy$ is not defined if either $x = 0$ or $y = 0$.

Complex fractions have numerators and denominators which themselves contain fractions. For example:

$$\frac{x/y}{(1/x) - (1/y)}.$$

Complex fractions can be reduced to simpler fractions through use of the rules given in this section.

EXAMPLE 3–1.

$$\frac{x/y}{(1/x) - (1/y)} = \frac{x/y}{(y - x)/xy} = \frac{x}{y} \cdot \frac{xy}{y - x} = \frac{x^2}{y - x}.$$

This fraction is not defined for $x = 0$, $y = 0$, or $x = y$.

To add

$$\frac{x^2 - y}{xy^2} + \frac{x + y^2}{x^2y},$$

the product x^3y^3 could be used as the common denominator, but it would be simpler to use x^2y^2 for this.

EXAMPLE 3–2.

$$\frac{x^2 - y}{xy^2} + \frac{x + y^2}{x^2y} = \frac{x^3 - xy}{x^2y^2} + \frac{xy + y^3}{x^2y^2} = \frac{x^3 + y^3}{x^2y^2}.$$

PROBLEM SET 3-2

1. Add the following fractions.

(a) $\frac{1}{x}, \quad \frac{-1}{y}$ (b) $\frac{3}{x}, \quad \frac{-x}{3}$ (c) $\frac{a}{b}, \quad \frac{b}{a}$

(d) $\frac{-3x}{y^2}, \quad \frac{9x^3}{y}$ (e) $\frac{-2}{x + 1}, \quad \frac{2}{x - 1}$

2. Multiply the fractions given in problem 1.

3. Divide the first fraction by the second in each part of problem 1.

4. Add the following sets of fractions.

(a) $\frac{6}{a^2}, \quad \frac{a}{9}, \quad \frac{-12}{a}$

(b) $\frac{1}{x}, \quad \frac{-1}{y}, \quad \frac{1}{xy}$

(c) $\frac{3}{x}, \quad \frac{-x}{3}, \quad \frac{1}{3x}$

(d) $\frac{-3x}{y^2}, \quad \frac{9x^3}{y}, \quad \frac{-2y^2}{x^4}$

(e) $\frac{2}{x+1}, \quad \frac{2}{x-1}, \quad \frac{-4}{x}$

5. Find the product of each of the sets of fractions in problem 4.

6. Divide the product of the first and second fractions by the third fraction for each of the sets of fractions in problem 4.

7. Simplify the following complex fractions and indicate the values of the variables for which the fraction is not defined. After the simplification is completed, use the auxiliary condition given in parentheses as a check.

(a) $\dfrac{\frac{a}{b} - \frac{b}{a}}{\frac{a}{b} + \frac{b}{a}}, \quad (a = b)$

(b) $\dfrac{1 + \frac{x}{x+y}}{1 - \frac{x}{x+y}}, \quad (x = y)$

(c) $\dfrac{x - \frac{x^2}{x-y}}{y + \frac{y^2}{x-y}}, \quad (y = 2x)$

(d) $\dfrac{\frac{2a}{a+1} - 2}{\frac{2}{a-1} + 2}, \quad (a = 3)$

(e) $\dfrac{\frac{1}{x} - \frac{1}{xy} + \frac{1}{x^2}}{\frac{1}{y} - \frac{1}{xy} + \frac{1}{y^2}}, \quad (x = y)$

3–6 Division as successive subtraction. Remainder. Just as multiplication with integers can be interpreted as successive additions (Section 1–9), so can division of integers be interpreted as successive subtractions. This is the method for performing such arithmetical operations on a computing machine. Division in arithmetic is such a process which combines the principle of place value, the distributive law of multiplication, and the subtractive process. Here, for example, instead of subtracting the divisor 8 times, you first multiply the divisor by 8 and then subtract, taking place value into account. This is illustrated by the following numerical example:

EXAMPLE 3–3.

```
  135
73)9876
   73
  257
  219
  386
  365
   21
```

In the first operation, subtract 73 from 98. In the second operation, subtract 73 from 257 as many times as possible (three times). Instead of subtracting 73 three times, 73 is multiplied by 3 and then subtracted. In the final operation, five times 73 is subtracted from 386.

The sum and product of two natural numbers is itself a natural number, but the quotient of two natural numbers need not be a natural number. In the above example, there is a remainder 21, and when the equation is written in terms of subtraction, we have

$$9876 - 73 \cdot 135 = 21.$$

If A and B are natural numbers, the general relation can be written

$$A - BQ = R, \tag{3–8}$$

where Q is a natural number (or zero) and R is a natural number (or zero) less than B. The number Q is called the *partial quotient* to distinguish it from the *complete* quotient A/B, and R is called the *remainder*. Equation (3–8) can be written in the equivalent forms

$$A = BQ + R, \quad \frac{A}{B} = Q + \frac{R}{B}. \tag{3–9}$$

The first of these forms is often stated:

Dividend = *Divisor* × *Partial Quotient* + *Remainder*.

The second of these forms is often referred to as an *improper fraction*. In Example 3–3,

$$9876 = 73 \cdot 135 + 21 \qquad \text{or} \qquad \tfrac{9876}{73} = 135\tfrac{21}{73}.$$

3–7 Decimal fractions. Definition and notation. A fraction whose denominator is a power of 10 is called a *decimal fraction*. Such fractions can be expressed in several ways:

(1) $\dfrac{A}{10^k}$,

where A is an integer and k is a positive integer;

(2) $B + \dfrac{a_1}{10} + \dfrac{a_2}{10^2} + \dfrac{a_3}{10^3} + \cdots + \dfrac{a_k}{10^k}$,

where B is a whole number, $a_1, a_2, \ldots, a_k$ are single digit numbers 0, 1, 2, . . . , 9, and k is a positive integer; or by use of the decimal point and the principle of position to write the fraction as

$$(3)\quad B\,.\,a_1a_2\cdots a_k.$$

For example, $\frac{7368}{1000} = 7.368 = 7 + \frac{3}{10} + \frac{6}{100} + \frac{8}{1000}$.

The number of decimal places in a fraction may continue indefinitely, thus giving a nonterminating decimal fraction. When such fractions repeat their digits in a periodic fashion, this is indicated by a dot over the first and last digit of the period. For example: 3.24727272 . . . , a periodic nonterminating decimal fraction, would be written as $3.24\dot{7}\dot{2}$. If a nonterminating decimal fraction is not periodic, a rule is needed to show how the sequence of digits is formed. Thus, the nonterminating, nonperiodic, decimal fraction .12345678910111213 . . . is formed by writing (after the decimal point) the natural numbers in order. The nonterminating, nonperiodic, decimal fraction .10110111011110 . . . is formed by writing after the decimal point a 1 followed by 0, then two 1's followed by 0, three 1's followed by 0, and so on, increasing the number of 1's by one each time.

Arithmetical operations. The arithmetical operations with decimal fractions are essentially the same as with any fractions.

To add (or subtract) decimal fractions, align all the decimal points to preserve proper place values and add (or subtract) as with integers.

EXAMPLE 3–4. $23.78 \pm 3.892 + 0.65$ could be computed as follows:

23.78	23.780
+3.892	−3.892
27.672	19.888
+0.65	+0.65
28.322	20.538

The rules for multiplying and dividing two decimal fractions are based on the usual rules for multiplying or dividing fractions and the laws of positive exponents.

$$\frac{A}{10^k} \times \frac{B}{10^m} = \frac{AB}{10^{k+m}},$$

where A and B are integers and k and m are positive integers.

$$\frac{A}{10^k} \div \frac{B}{10^m} = \frac{A}{10^k} \cdot \frac{10^m}{B} = \begin{cases} \dfrac{A}{B} \cdot 10^{m-k} & \text{if} \quad m > k, \\ \dfrac{A}{B} \div 10^{k-m} & \text{if} \quad m < k. \end{cases}$$

To multiply two decimal fractions, multiply as with integers; the number of decimal places in the product is the sum of the numbers of decimal places in the factors.

To divide two decimal fractions, divide as with integers; if the number of decimal places used in the dividend is greater than or equal to the number of decimal places in the divisor, the number of decimal places in the quotient is their difference; if the number of decimal places in the divisor is greater than the number of decimal places in the dividend, then add zeros as decimal places to the dividend in sufficient number to give it as many such places as the divisor.

EXAMPLE 3–5. These rules are illustrated below by the division of 0.7325 by 6.89 and 7325.0 by 6.89.

```
      .106          6.89
6.89).73250         .106
      689           4134
      4350         6890
      4134         .73034
       216         .00216
                   .73250   check.
```

Since five decimal places are used in the dividend and two in the divisor, there are three decimal places in the quotient. An advantage of this method is that it preserves the proper position values for the remainder:

$$0.7325 = (0.106)(6.89) + 0.00216.$$

For the second division, we have

```
      1063
6.89)7325.00
      689
      43500
      4134
       2160
       2067
         93
```

Since the divisor uses two decimal places, and the dividend only one, a zero is added to the decimal part of the dividend. The corresponding equation is $7325 = (1063)(6.89) + 0.93$.

Decimal fractions and common fractions. A positive terminating decimal is equivalent to a common rational fraction of the form $A_1/(2^p5^q)$, where A_1, p, q are positive integers. If the given decimal fraction is $A/10^k$, the only integral factors of 10 are 2 and 5, and A may be divisible by some power of 2 and by some power of 5. Conversely, any rational fraction of the form $A_1/(2^p5^q)$ is equivalent to a decimal fraction. If $p = q$, it is already in such a form: $A_1/10^p$; if $q > p$, multiply both numerator and denominator by 2^{q-p} to obtain

$$\frac{A_1}{2^p5^q} \cdot \frac{2^{q-p}}{2^{q-p}} = \frac{A}{10^q},$$

a decimal fraction. If $p > q$, multiply both numerator and denominator by the appropriate power of 5:

$$\frac{A_1}{2^p5^q} \cdot \frac{5^{p-q}}{5^{p-q}} = \frac{A}{10^p},$$

a decimal fraction.

If the rational fraction, in its lowest terms, is not of the form above, it is equivalent to a periodic nonterminating decimal fraction. Conversely, every periodic, nonterminating decimal fraction is equivalent to a common fraction. If a/b is such a fraction, divide by the successive subtraction process, annexing as many zeros after the decimal point as desired. In this process, one eventually comes to a place where only 0's are brought down from the dividend. Further, at any step in the process, the remainder (disregarding the decimal point) must always be less than b. If this remainder is zero, the process stops and a terminating decimal fraction results. This happens if and only if b is of the form 2^p5^q; otherwise, the remainder is between 0 and b, and since there are at most $(b - 1)$ such integers, a place is reached where the remainder repeats and where zeros have been brought down from the dividend in both cases. When this happens, the fraction begins to repeat. Some simple examples are $\frac{1}{3} = 0.\dot{3}$, $\frac{1}{7} = 0.\dot{1}42857\dot{}$, $\frac{1}{11} = 0.\dot{0}\dot{9}$. The general process is now illustrated by the fraction $\frac{893}{275}$. Although the denominator contains the factor 5, it also contains the factor 11, so the decimal equivalent is periodic.

EXAMPLE 3-6.

$$
\begin{array}{r}
3.2472 \\
275\overline{)893.0000} \\
825 \\
\hline
680 \\
550 \\
\hline
1300 \\
1100 \\
\hline
2000\ * \\
1925 \\
\hline
750 \\
550 \\
\hline
2000^*
\end{array}
\qquad \frac{893}{275} = 3.247\dot{2}
$$

* The remainder now repeats and the period is determined.

For a periodic decimal fraction with a nonperiodic part, the nonperiodic part is equivalent to a rational fraction, so it suffices to prove that the periodic part is equivalent to a rational fraction, and then to take the sum of the two parts. By multiplying and dividing by an appropriate power of ten, the periodic decimal fraction can be made to begin with the first decimal place without affecting the rational property.

EXAMPLE 3-7.

To prove $3.247\dot{2}$ represents a rational fraction, write

$$3.247\dot{2} = 3.24 + \tfrac{1}{100}(.\dot{7}\dot{2})$$

and consider the fraction $0.727\dot{2}\dot{7}\dot{2}\ldots$ If we call it x, then

$$x = 0.727\dot{2}\dot{7}\dot{2}\ldots$$

$$100x = 72.727272\ldots = 72 + x$$

$$99x = 72; \quad x = \frac{77}{99} = \frac{8}{11}.$$

Hence
$$3.24\dot{7}\dot{2} = \frac{324}{100} + \frac{1}{100}\cdot\frac{8}{11} = \frac{324 \times 11 + 8}{1100}$$

$$= \frac{3572}{1100} = \frac{893}{275}.$$

In order to have uniqueness in the decimal expression for a fraction, it is understood that an unending sequence of 9's is equivalent to increasing the digit before the first 9 by 1. This follows from $0.9\dot{9}\dot{9}\ldots = 1$, which is proved as follows:

Let
$$x = 0.9\dot{9}\dot{9}\ldots$$

$$10x = 9.9\dot{9}\ldots = 9 + x$$

$$9x = 9$$

$$x = 1.$$

In general, suppose that the decimal fraction consists of a rational fraction and a periodic part which is represented by

$$\frac{1}{10^p}(.\dot{a}_1 a_2 \ldots \dot{a}_k) = \frac{1}{10^p}\left(\frac{A}{10^k} + \frac{A}{10^{2k}} + \frac{A}{10^{3k}} + \cdots\right),$$

where the a's are single digit numbers and A is the integer represented by these digits. Designate the fraction $(.\dot{a}_1 a_2 \ldots \dot{a}_k)$ by x:

$$x = .\dot{a}_1 a_2 \ldots \dot{a}_k = A\left(\frac{1}{10^k} + \frac{1}{10^{2k}} + \frac{1}{10^{3k}} + \cdots \text{forever}\right).$$

Then

$$10^k x = A + A\left(\frac{1}{10^k} + \frac{1}{10^{2k}} + \cdots \text{forever}\right)$$

$$= A + x$$

$$(10^k - 1)x = A$$

$$x = \frac{A}{10^k - 1},$$

which is a rational fraction. Dividing x by 10^p and adding the nonrepeating part will not affect the property of being rational. This proof gives a procedure for finding the rational fraction which is equivalent to a periodic decimal fraction, as illustrated by the numerical example above.

Decimal approximations. Often decimal fractions are replaced by decimal approximations to these numbers. Experimental data is subject to either systematic or random error, and in most problems of measurement only approximate results are obtained. The decimal forms of rational numbers, even when the whole period is known, are often too long to be of practical value. When irrational, their decimal forms neither terminate nor repeat, so decimal approximations must be used. For example, tables of square roots and cube roots of numbers give decimal approximations only. A table of squares might give $(3.47)^2 = 12.04$ but this does not mean that 12.04 is exact. Actually, $(3.47)^2 = 12.0409$. From Table I, $\sqrt{3.47}$ is 1.863, but this result is only approximately correct; indeed $\sqrt{3.47} = 1.86279^+$.

The number of *significant figures* in a decimal fraction is the number of digits used in writing the number, exclusive of those merely used to locate the decimal point. Thus 12.04, 0.01204, 0.1240 all have four significant figures. When an integer terminates with one or more zeros, the final zeros may or may not be significant. Thus 7325/6.89 = 1060., approximately. The final zero is used to locate the decimal point and is not significant. If the division is carried further, we find 7325/6.89 = 1063., the final 3 being significant.

To *round off* a given decimal fraction to N significant figures means to write the decimal fraction retaining only N significant figures. This is done by use of the *computer's rule:*

If the final digit is less than 5, it is dropped; if it is greater than 5, the digit which precedes it is increased by 1; if the final digit is 5 and nothing more is known about the number, the last digit retained is made even.

EXAMPLE 3–8. The number 3.1415965 rounded off successively is 3.141596, 3.14160, 3.1416, 3.142, 3.14. The number 0.00431746 rounded off successively is 0.0043175, 0.004317, 0.00432. The final fraction has three significant figures.

In working with numerical calculations involving decimal fractions taken from tables or computed by special devices, it is important to have working rules which control the accuracy of the final result. Over-calculation due to using too many significant figures and retaining more significant figures than are justified by the data should be avoided. On the other hand, under-calculation due to not using as many significant figures as are justified by the data should also be avoided. The principle underlying calculation rules is that the result cannot be more accurate than is dictated

by the accuracy of the least accurate of the numbers used. Numerical calculations by addition and subtraction depend essentially on the number of decimal places used, while those for multiplication and division depend essentially on the number of significant figures used.

In performing additions and subtractions, round off all numbers to as many decimal places as are significant in the number with the least number of significant decimal places.

If several numbers are to be added, it is advisable to make the initial round off retain one more decimal place than is indicated by the above rule and then round off the final answer once more.

EXAMPLE 3-9. If it is desired to add $(2.16)^2$ and $(3.47)^2$, using a four-place table of squares which gives these as 4.666 and 12.04, the addition is written

$$\begin{array}{r} 4.67 \\ 12.04 \\ \hline 16.71 \end{array}$$

where the result is correct to four significant figures.

In multiplication (*and division*), round off all numbers, retaining the number of significant figures found in the number with the least number of significant figures and then round off the product (or quotient) to this same number of significant figures.

EXAMPLE 3-10. Compute $\sqrt{23} \times \sqrt{40}$ and $\sqrt{23} \div \sqrt{40}$.

A four-place table of square roots (Table I, Appendix III), gives

$$\sqrt{23} = 4.796 \quad \text{and} \quad \sqrt{40} = 6.325.$$

The product $(4.796)(6.325) = 30.334700$, and hence

$$\sqrt{23}\,\sqrt{40} = 30.33, \text{ approximately.}$$

For the division:

$$\begin{array}{r} .7581 \\ 6.325\overline{)4.7960000} \\ 4\,4275 \\ \hline 36850 \\ 31625 \\ \hline 51250 \\ 50600 \\ \hline 6500 \end{array}$$

Hence $\sqrt{23} \div \sqrt{40} = 0.7581$, approximately.

EXAMPLE 3–11. To compute $(2.16)^2 + (2.16)(3.47) + (3.47)^2$, two round-off procedures are given:

		rounding off first
$(2.16)^2 =$	4.666	4.67
$(3.47)^2 =$	12.04	12.04
$(2.16)(3.47) =$	7.495	7.50
Sum =	24.201	24.21
=	24.20 approx.	

PROBLEM SET 3–3

1. Express each of the following decimal fractions as a common fraction in its lowest terms.

(a) 0.064 (b) 2.48 (c) 0.675 (d) 3.04

2. Explain why each of the following common fractions is equivalent to a terminating decimal fraction and find the decimal fraction in two ways: (1) by multiplying numerator and denominator by the number which will make the denominator a power of ten; and (2) by performing the indicated division.

(a) $\frac{5}{64}$ (b) $\frac{9}{625}$ (c) $\frac{73}{40}$ (d) $\frac{57}{160}$

3. Write some nonperiodic, nonterminating, decimal fractions and explain in words the rule for their formation.

4. If it were desired to prove that the decimal equivalent of $\sqrt{3}$ would never terminate or repeat, what must be proven about the number $\sqrt{3}$?

5. Find the common fractions in lowest terms which are equivalent to the following repeating decimal fractions.

(a) $.42\dot{6}\dot{9}$ (b) $2.0\dot{9}\dot{9}$ (c) $0.\dot{5}\dot{5}$, $0.3\dot{5}\dot{5}$, $4.3\dot{5}\dot{5}$

(d) $0.\dot{5}\dot{7}$, $0.4\dot{5}\dot{7}$, $3.4\dot{5}\dot{7}$ (e) $4.\dot{3}5\dot{6}$

6. Express each of the following fractions as a decimal fraction. Find six decimal places for the answer and then round off this to four significant figures.

(a) $\frac{6}{11}$ (b) $\frac{13}{37}$ (c) $\frac{4}{13}$ (d) $\frac{5}{13}$ (e) $\frac{3}{7}$ (f) $\frac{3}{17}$

7. If the values of $\sqrt{7}$, π, and e are taken from tables to be approximately 2.646, 3.1416, and 2.7183, respectively, find the following sums, products, and quotients as accurately as is consistent with this data.

(a) $\sqrt{7} + \pi + e$ (b) $\pi\sqrt{7}$ (c) πe

(d) π/e (e) $\sqrt{7}/e$

8. If the approximate values of the powers of e taken from a table are $e = 2.7183$, $e^2 = 7.3891$, $e^3 = 20.086$, $e^4 = 54.598$, compute $e + e^2 + e^3 + e^4$, as accurately as is consistent with this data.

9. If the following data is obtained from a table of squares: $(3.46)^2 = 11.97$; $(1.77)^2 = 3.133$, compute

(a) $(3.46)^2 + (1.77)^2$ (b) $(3.46)^2 + 3(3.46)(1.77) + (1.77)^2$

(c) $(3.46)^2(1.77)^2$ (d) $(1.77)^2/(3.46)^2$

as accurately as is consistent with this data.

3–8 Percentage and applications. If the denominator of a fraction is 100, the fraction is expressed in per cent and this is indicated by using the symbol % instead of the denominator. The numerator may be an integer, an improper fraction, or a decimal fraction. $N/100$ and $N\%$ are equivalent expressions. Thus

$$0.15 = \frac{15}{100} = 15\%, \qquad \frac{1}{6} = \frac{16\frac{2}{3}}{100} = 16\tfrac{2}{3}\%,$$

$$0.0625 = 6.25\% = 6\tfrac{1}{4}\%, \qquad \frac{7}{12}\% = \frac{7}{1200} = 0.005833 = 0.5833\%$$

to four significant figures. Since the change from a decimal fraction to per cent is merely a process of moving the decimal point two places to the right, no essentially new concepts are involved.

If P and B are any two numbers, the ratio of P to B is given by:

$$r = \frac{P}{B}, \quad P = rB, \quad B = \frac{P}{r}. \tag{3–10}$$

If r is expressed as a decimal fraction, then

$$r = \frac{100r}{100} = (100r)\%.$$

In financial transactions, r is known as the rate, P as the Percentage, and B as the Base, and Eq. (3–10) is written as

$$r(\text{ate}) = \frac{P(\text{ercentage})}{B(\text{ase})}.$$

If any two of the three numbers r, P, B are given, then the other is directly determined by Eq. (3–10). Ordinarily the Percentage and Base are determined to the nearest cent, while the rate is given to the proper number of significant figures which justify giving the Percentage to the nearest cent. Financial transactions of a number of types are now illustrated.

(a) *Commission.* A salesman often receives his income as a stated per cent of the selling price (Base), $C = rB$. If two of the quantities r, B, C are given, the third can be found.

EXAMPLE 3-12. An insurance policy salesman receives 45% of the first premium and 12½% of the second and third premiums. If the annual premium on a given policy is \$78.63, what is his income from such a sale?

$$C = \$78.63(45\% + 12\tfrac{1}{2}\% + 12\tfrac{1}{2}\%) = \$78.63(70\%) = \$55.04.$$

He receives \$35.38 the first year and \$9.83 for each of the other years.

EXAMPLE 3-13. A car salesman offers to sell a car for \$1650, which he claims is 15% below its true value. What is the claimed true value to the nearest dollar?

$$\$1650 = 85\%V \quad \text{or} \quad V = \frac{\$1650}{0.85} = \$1941.$$

(b) *Markup and markdown.* A retailer who purchases articles from a wholesaler at stated prices expects to sell the articles at a price which will pay his expenses and provide a profit. This is often done by marking up the wholesale price by a fixed per cent, say $(100r)\%$.

$$\text{Markup} = r \times \text{Wholesale Price} \quad (M = rW).$$

$$\text{Selling Price} = \text{Wholesale Price} + \text{Markup} \quad (S = W + M).$$

In order to move the merchandise more rapidly, he may at some future date offer an article at a marked-down price, where the markdown is a stated per cent, say $(100s)\%$ of the selling price.

$$\text{Markdown} = s \times \text{Selling Price} \quad (D = s \cdot S)$$

$$\text{Sales Price} = \text{Selling price} - \text{Markdown} \quad S_1 = S - D.$$

(c) *Profit and loss.* The rate of profit is determined by

$$r = \frac{\text{Profit}}{\text{Base}},$$

where it is necessary to state what the base is. It might be the wholesale price, or the wholesale price plus the expense of selling, or it might be the selling price, with the profit determined with or without the expenses taken into account. A problem is not determined until the base is explicitly described.

EXAMPLE 3-14. A retailer pays \$8.50 for an article and marks up this price by 25%, rounded off to the nearest dime. (a) What per cent of the selling price is this markup? (b) If the estimated cost of selling the article is 10% of the wholesale price, what is his profit, and what per cent of the wholesale price and the retail price is this profit?

(a) Markup = 25% of \$8.50, rounded off to one decimal place,

= \$2.10

Selling price = \$10.60.

It would be meaningless to carry this calculation further, since 2.10 is given to only three significant figures.

(b) Profit = \$10.60 − \$8.50 − \$0.85 = \$1.25

$$r_1 = \frac{1.25}{8.50} = 14.7\%; \quad r_2 = \frac{1.25}{10.60} = 11.8\%.$$

Neither of these rates takes into account the retailer's total investment, which should include the wholesale price plus the selling cost. If this total investment is used as the base, then the rate of profit is $r_3 = 1.25/9.35 = 13.4\%$ to three significant figures.

(d) *Trade discounts.* In order to avoid frequent publication of price lists, to stabilize or control retail prices, or to adjust prices to a changing economy, the manufacturer or wholesaler issues catalogues of so-called list prices. The retailer receives a separate discount sheet which can be changed frequently and which shows the discount in per cent allowed to the retailer. This discount is often quoted in terms of successive discounts, where the first discount is based upon the list price, the second discount is based upon the first discounted price, and so on. Where not prohibited by "fair trade" laws, the retailer may mark down the list prices to facilitate sales.

EXAMPLE 3–15. A catalogue lists the price of a television set at \$250. The dealer is allowed successive trade discounts of 35%, 10%, and 5%. What does he pay for the set? What single trade discount is equivalent to these successive discounts?

Discount = (rate)(Base); Net Price = List Price − Discount.

$D_1 = 0.35 \times \$250 = \$87.50$, $B_1 = \$250.00 - \$87.50 = \$162.50$

$D_2 = 0.10 \times \$162.50 = \$16.25$, $B_2 = \$162.50 - \$16.25 = \$146.25$

$D_3 = 0.05 \times \$146.25 = \$7.31$ Price = \$146.25 − \$7.31 = \$138.94

Total Discounts = \$111.06 = r(\$250),

or

$$r = \frac{111.06}{250} = 44.42\%, \text{ approximately. Ans.}$$

(e) *Taxes.* Certain taxes, like those collected on real estate, are based upon the assessed value of the property and a fixed rate. Income taxes are based upon net income (income after all exemptions are deducted) and upon variable rates, depending upon the amount of net income.

EXAMPLE 3–16. A certain piece of property was assessed for \$6040 and the county taxes were \$488.72, but the tax rate was not published. What was the tax rate?

$$r = \frac{488.72}{6040.} = 8.0914\%.$$

In order to justify the tax to the nearest cent, it is necessary to compute the rate to five significant figures.

EXAMPLE 3–17. The U.S. government income tax on a net income of \$14,240 is 20% of the first \$4000, 22% of the next \$4000, 26% of the next \$4000, and 30% of the amount which exceeds \$12,000. Find the amount of the tax and an equivalent single tax rate.

$$T = \$4000\ (20\% + 22\% + 26\%) + \$2240\ (30\%)$$
$$= \$3392$$
$$r = \frac{3392}{14240} = 23.82\%.$$

EXAMPLE 3–18. A certain stock has paid a regular annual dividend (Percentage) of \$3.50 a share. How much should you pay for the stock (Base) if you wish to earn 6% on your investment?

$$B = \frac{\$3.50}{.06} = \frac{\$350}{6} = \$58.33.$$

PROBLEM SET 3–4

1. A salesman receives a commission of 7% of his daily sales up to \$100, and 4% of the daily sales over \$100. If his daily sales average \$280, what is his average daily commission?

2. An insurance policy salesman receives 30% of the first premium on a certain policy and 6% of the next four premiums. If the annual premium is \$68.72, what is his total income from such a sale?

3. The assessed value of a piece of property is \$4760. The tax rate is 7.5842%. What is the amount of the tax?

4. An article cost the retailer \$24; he estimates his overhead (selling expenses) as 12% of this cost. (a) If he desires to make a profit of 10% on his cost, what should the selling price be? (b) If he sells the articles for \$30, what % of his cost is his profit?

12398

5. (a) \$47.50 is what per cent of \$763?
(b) 27.6 is what per cent of 23.9?
(c) \$45.50 is what per cent of \$6000?

6. A retailer pays \$17.25 for an article and marks up this price by $33\frac{1}{3}\%$, rounded off to the nearest dime (a) What per cent of the selling price is this markup? (b) If the estimated cost of selling the article is 12% of the cost price, what is his profit? (c) What per cent of (i) the cost, (ii) the cost plus the selling cost, (iii) the selling price is this profit?

7. A dealer pays \$4.50 for a tool and marks up the price $33\frac{1}{3}\%$. At a sale he offers this tool at 15% off the original selling price. If his selling cost averages \$0.50 per tool, what per cent of his total costs does he gain or lose?

8. An article whose list price is \$15.75 is sold to a retailer at successive discounts of 10%, 7%, 3%. (a) What does the retailer pay for the article? (b) What single discount is equivalent to these three successive discounts?

9. A catalogue lists the price of a radio at \$160. The dealer is allowed successive discounts of 25%, 10%, and 5%. (a) What does he pay for the radio, and what single trade discount would yield the same cost? (b) The dealer's overhead amounts to 20% of the list price. At a later date, he offers a customer the set at 15% off list price. Does he lose or gain on the whole transaction?

10. On a recent tax bill, a piece of property was assessed for \$6480 and the taxes stated as \$498.78, but the tax rate was omitted. What was the tax rate?

11. The U.S. government graduated income tax rates in 1958 were 20% of net incomes less than \$4000; 22% of the net income between \$4000 and \$8000; 26% of the net income between \$8000 and \$12,000; 30% of the net income between \$12,000 and \$16,000. What is the amount of the tax on the following net income: (a) \$7750; (b) \$10,760; (c) \$14,820?

12. (a) 19.5 is 28% of what number?
(b) 76.3 is 104% of what number?
(c) 3.76 is .0342% of what number?

13. A dress selling for \$13.75 is marked "15% off list price." What was the list price?

14. A car is offered for sale for \$1450. The salesman claims that this is 18% off its true value. What is the claimed true value?

15. A certain stock has paid regular annual dividends of \$6.25 for a number of years. How much should one expect to pay for the stock to earn (a) 7%, (b) 5% on his investment?

16. A bond salesman earns a commission of $1\frac{1}{2}\%$ of his sales. How much must he sell a month to have an income of \$800 per month?

17. A retailer collects a 4% sales tax on all his sales, but failed to keep the sales tax money separated from the selling price of the goods. At the end of the day, his total receipts were \$458.64. How much of this was tax money? Is this amount less than or greater than 4% of \$458.64? Explain.

3–9 Factoring. A natural number is factored with regard to the natural numbers by finding two or more natural numbers (omitting +1) whose product is this number. For example, $14 = 2 \cdot 7$. To factor 14 over the set of integers, we also have $14 = (-2)(-7)$.

To factor a polynomial in one (or more variables), one first specifies the set of numbers to which the coefficients in the factors belong. This section treats factoring of polynomials with rational coefficients over the set of rational numbers, and the special case of factoring polynomials with integral coefficients over the set of integers. To factor a polynomial over the set of integers means to find two or more polynomials with integral coefficients whose product is the given polynomial. Since the coefficients of the product of two such polynomials involve only the addition and multiplication of integers, the coefficients of the original polynomial must be integers. To factor a polynomial over the set of rational numbers means to find two or more polynomials with rational coefficients whose product is the given polynomial, and this implies that the coefficients in the original polynomial must be rational numbers. Thus $x^2 - 2x + \frac{21}{25}$ could not be factored over the set of integers, but can be factored over the set of rational numbers:

$$x^2 - 2x + \tfrac{21}{25} = (x - \tfrac{3}{5})(x - \tfrac{7}{5}) \quad \text{for any } x.$$

The polynomial $x^2 - 4$ could be written as the product of two polynomials in many ways:

$$x^2 - 4 = (x - 2)(x + 2) = -(2 - x)(2 + x) = (2x - 4)\left(\frac{x}{2} + 1\right).$$

The first two factorizations are over the set of integers. The third equality is one of many possible factorizations over the set of rational numbers. The polynomial $x^2 - 2$ can be written in the form

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2}),$$

but the factors do not have rational ceofficients. $x^2 - 2$ is a simple illustration of a polynomial which cannot be factored over the set of rational numbers, but which can be factored over the set of all real numbers; $x^2 + 2$ cannot be factored over the set of all real numbers.

If a polynomial has rational coefficients, it may be treated as a polynomial with integral coefficients which has been divided by some constant. Hence the methods for finding the factors over the set of rational numbers can be reduced to that of finding the factors of a polynomial with integral coefficients over the set of integers. Thus

$$\begin{aligned} x^2 - 2x + \tfrac{21}{25} &= \tfrac{1}{25}(25x^2 - 50x + 21) \\ &= \tfrac{1}{25}(5x - 3)(5x - 7) \\ &= (x - \tfrac{3}{5})(x - \tfrac{7}{5}) = (\tfrac{3}{5} - x)(\tfrac{7}{5} - x). \end{aligned}$$

The methods for finding the factors of a polynomial with integral coefficients over the set of integers are discussed under several subheadings.

(A) *Special products.* Any of the special products given in Section 2–6, except possibly Eqs. (2–11) and (2–12), can be read directly from right to left to get formulas for factoring.

EXAMPLE 3–19. Thus one recognizes immediately that

$$\tfrac{4}{9}x^2 - \tfrac{9}{4}y^2 = (\tfrac{2}{3}x - \tfrac{3}{2}y)(\tfrac{2}{3}x + \tfrac{3}{2}y),$$

which gives the factors over the set of rational numbers.

(B) *Grouping.* The distributive law of multiplication (M8), read from right to left, can often be used to factor certain algebraic expressions by an appropriate grouping of terms.

EXAMPLE 3–20.

$$\begin{aligned} ax + ay + bx + by &= a(x + y) + b(x + y) \\ &= (a + b)(x + y), \end{aligned}$$

or

$$\begin{aligned} ax + ay + bx + by &= x(a + b) + y(a + b) \\ &= (a + b)(x + y). \end{aligned}$$

EXAMPLE 3–21. Factor $2x^3 - 2y^3 - x^2y + 4xy^2$. Group the first term with the third and the second term with the fourth to get

$$\begin{aligned} 2x^3 - 2y^3 - x^2y + 4xy^2 &= x^2(2x - y) + 2y^2(2x - y) \\ &= (2x - y)(x^2 + 2y^2). \end{aligned}$$

(C) *Trial factors.* Equations (2–11) and (2–12) provide methods for factoring some polynomials of the form $Ax^2 + Bx + C$ and $Ax^2 + Bxy + Cy^2$, where A, B, C are integers. Thus:

$$Ax^2 + Bx + C = (cx + a)(dx + b)$$

or

$$Ax^2 + Bxy + Cy^2 = (cx + ay)(dx + by),$$

where a, b, c, d are integers such that $cd = A$, $ab = C$, $ad + bc = B$. There are at most a finite number of pairs of integers (a, b) and a finite number of pairs of integers (c, d) to try.

EXAMPLE 3–22. Factor

(a) $x^2 + 4x + 3$; (b) $x^2 - x - 6$; (c) $15x^2 - 11xy - 14y^2$

over the set of integers.

(a) To factor $x^2 + 4x + 6$, we limit the trials to the positive integer pairs $a = 1$, $b = 6$; $a = 2$, $b = 3$, for interchanging a and b would not affect the result. Since neither $1 + 6$ nor $2 + 3$ is 4, $x^2 + 4x + 6$ is not factorable over the positive integers.

(b) To factor $x^2 - x - 6$ into $(x + 2)(x - 3)$, we may limit the trials to integer pairs $a = 1$, $b = -6$; $a = -1$, $b = 6$; $a = 2$, $b = -3$; $a = -2$, $b = 3$.

(c) To factor $15x^2 - 11xy - 14y^2$, we need to try the pairs (1, 15), (3, 5) as the coefficients of x. Each of these is combined with each of the following pairs of possible coefficients for y:

$(1, -14); (-1, 14); (2, -7); (-2, 7); (7, -2); (-7, 2); (14, -1); (-14, 1)$.

Hence there are sixteen pairs of factors to try. They are listed below followed by the coefficient of xy.

$(x + y)(15x - 14y)$	(1)	$(3x + y)(5x - 14y)$	(-37)
$(x - y)(15x + 14y)$	(-1)	$(3x - y)(5x + 14y)$	(37)
$(x + 2y)(15x - 7y)$	(23)	$(3x + 2y)(5x - 7y)$	(-11)
$(x - 2y)(15x + 7y)$	(-23)	$(3x - 2y)(5x + 7y)$	(11)
$(x + 7y)(15x - 2y)$	(103)	$(3x + 7y)(5x - 2y)$	(29)
$(x - 7y)(15x + 2y)$	(-103)	$(3x - 7y)(5x + 2y)$	(-29)
$(x + 14y)(15x - y)$	(209)	$(3x + 14y)(5x - y)$	(67)
$(x - 14y)(15x + y)$	(-209)	$(3x - 14y)(5x + y)$	(-67)

It follows that $15x^2 - 11xy - 14y^2 = (3x + 2y)(5x - 7y)$ for all x and y. It also follows that $15x^2 + Bxy - 14y^2$ can be factored over the set of integers for the following values of B:

$$\pm 1, \quad \pm 23, \quad \pm 103, \quad \pm 209, \quad \pm 37, \quad \pm 11, \quad \pm 29, \quad \pm 67.$$

Problem Set 3-5

When possible, factor each of the following into its factors of lowest degree over the set of integers or over the rational numbers. Otherwise, show that such factorization is impossible.

1. $4x^2 - 9y^2$
2. $a^2 - 6ab + 9b^2$
3. $4x^2 + 12xy + 9y^2$
4. $\frac{4}{9}x^2 - 2xy + \frac{9}{4}y^2$
5. $27x^2 - 48y^2$
6. $2a^3 - 8a^2b + 8ab^2$
7. $x^3 - 27y^3$
8. $x^3 + 9x^2y + 27xy^2 + 27y^3$
9. $x^6 - 64y^6$
10. $x^6 + 64y^6$

11. $3ab - 6bx - 4ay + 8xy$
12. $3ab + 6bx - 4ay + 8xy$
13. $5a^2 - 15ab + 6ac - 18bc$
14. $5a^2 - 15ab + 6ac - 12bc$
15. $3x^2 - 11x + 6$
16. $x^2 + \frac{11}{3}x + 2$
17. $x^2 - 2x + \frac{8}{9}$
18. $3x^2 + 2x - 8$
19. $3x^2 + 6x - 8$
20. $15x^2 + 22x + 8$
21. $12x^2 - ab - 20b^2$
22. $12a^2 - 35ab + 25b^2$
23. $15x^2 + 29xy - 14y^2$
24. $15x^2 + 27xy - 14y^2$

25. Justify on the basis of Eqs. (3-5), (3-6), (3-7) that the product, quotient, and sum of two rational numbers is a rational number.

26. Show that $x^2 - 2x - 4$ is not factorable over the set of integers. Does this imply that $x^2 - 2x - 4$ is not factorable over the set of rational numbers? Why? Does it imply that $x^2 - 2x - 4$ is not factorable over the set of real numbers? Explain.

3-10 Division of polynomials. The concepts of division as successive subtraction with a remainder for integers (see Section 3-6) are readily extended to division by a polynomial. By the distributive law of multiplication and the law for positive integral exponents, Eq. (2-5), the sum and product of two polynomials are also polynomials. To divide one polynomial by a second polynomial, subtract the divisor, multiplied by various powers of the variable, as many times as possible, the process terminating when a remainder whose degree is less than the degree of the divisor is reached.

EXAMPLE 3-23. In Section 2-6, we found

$$(x^2 - 2x + 3)(x^2 - x + 2) = x^4 - 3x^3 + 7x^2 - 7x + 6.$$

The division of this product by $x^2 - x + 2$, through the process of successive subtraction is given below:

$$\begin{array}{r} x^2 - 2x + 3 \\ \hline x^2 - x + 2 \;)\; x^4 - 3x^3 + 7x^2 - 7x + 6 \\ x^4 - x^3 + 2x^2 \\ \hline -2x^3 + 5x^2 - 7x \\ -2x^3 + 2x^2 - 4x \\ \hline 3x^2 - 3x + 6 \\ 3x^2 - 3x + 6 \\ \hline 0 \end{array}$$

Multiply $x^2 - x + 2$ by x^2 and subtract

Multiply $x^2 - x + 2$ by $-2x$ and subtract

Multiply $x^2 - x + 2$ by 3 and subtract.

The final remainder is zero.

EXAMPLE 3-24. Another example which illustrates the case where the remainder need not be zero and the coefficients need not be integers is:

$$
\begin{array}{rl}
 & 3x + \frac{1}{2} \\
2x^2 + x \,\overline{)}\, & 6x^3 + 4x^2 - 2x + 5 \\
 & \underline{6x^3 + 3x^2} \\
 & x^2 - 2x \\
 & \underline{x^2 + \frac{1}{2}x} \\
 & -\frac{5}{2}x + 5
\end{array}
$$

Multiply $2x^2 + x$ by $3x$ and subtract

Multiply $2x^2 + x$ by $\frac{1}{2}$ and subtract.

The remainder is $-\frac{5}{2}x + 5$, and

$$6x^3 + 4x^2 - 2x + 5 - (2x^2 + x)(3x + \tfrac{1}{2}) = (-\tfrac{5}{2}x + 5).$$

Symbols such as $F(x)$, $D(x)$, $Q(x)$, and $R(x)$ are used to represent polynomials in one variable x. The degrees of these general polynomials are represented by letters like n and m which denote positive integers.

If $F(x)$ and $D(x)$ are given polynomials of degree n and m, respectively, then the process of successive subtraction, corresponding to division of $F(x)$ by $D(x)$, is

$$F(x) - D(x)Q(x) = R(x), \quad \text{(for all } x\text{)}, \tag{3–11}$$

where $Q(x)$ is the partial quotient polynomial of degree $n - m$ when $n \geqq m$, and $R(x)$ is the remainder polynomial of degree less than m.

If $m > n$, $F(x) = R(x)$. Equation (3–11) can be written in the equivalent forms

$$F(x) = D(x)Q(x) + R(x),$$

$$\frac{F(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}. \tag{3–12}$$

In the last form, exclude all values of x for which $D(x) = 0$.

If $F(x)$ is a polynomial, then $F(a)$ is the result of evaluating this polynomial for the special number a. A division may be checked by assigning convenient values to x, avoiding those values of x for which $D(x) = 0$.

EXAMPLE 3–25. The division (see Example 3–24)

$$\frac{6x^3 + 4x^2 - 2x + 5}{2x^2 + x} = (3x + \tfrac{1}{2}) + \frac{-\frac{5}{2}x + 3}{2x^2 + x}$$

can be checked by using any value for x other than $x = 0$ or $x = -\frac{1}{2}$. For $x = 1$, the left member becomes $\frac{13}{3}$; the right member also becomes

$$\frac{7}{2} + \frac{5/2}{3} = \frac{21}{6} + \frac{5}{6} = \frac{26}{6} = \frac{13}{3}.$$

The case when $D(x)$ is a polynomial of the first degree, $D(x) = x - r$, is of special interest. Equation (3–12) then becomes

$$F(x) = (x - r)Q(x) + R, \quad \text{(for any } x\text{)}, \tag{3–13}$$

where r and R are now constants.

Since Eq. (3–13) is true for any x, it is true for the special value $x = r$, and thus $F(r) = R$.

REMAINDER THEOREM. *If a polynomial $F(x)$ is divided by $x - r$, the remainder is the result of evaluating $F(x)$ at $x = r$: $F(r) = R$.*

FACTOR THEOREM. *If a polynomial $F(x)$ is divided by $x - r$, the remainder is zero if and only if $x - r$ is a factor of $F(x)$.*

This follows directly from Eq. (3–13), where the statement $x - r$ is a factor of $F(x)$ means that it gives a complete quotient (remainder $R = 0$).

EXAMPLE 3–26. Determine whether $x - 2$ and $x + 2$ are factors of $x^3 - 6x - 4$. If so, find the other factor.

Let $F(x) = x^3 - 6x - 4$. Then $F(2) = 8 - 12 - 4 \neq 0$; $F(-2) = -8 + 12 - 4 = 0$. Hence $x + 2$ is one factor and the other is found by dividing $x^3 - 6x - 4$ by $x + 2$.

$$\begin{array}{r} x^2 - 2x - 2 \\ x + 2 \overline{)\, x^3 \qquad - 6x - 4} \\ \underline{x^3 + 2x^2} \\ \quad - 2x^2 - 6x \\ \quad \underline{- 2x^2 - 4x} \\ \qquad\quad - 2x - 4 \\ \qquad\quad \underline{- 2x - 4} \\ \qquad\quad\quad 0 \quad \text{Check.} \end{array}$$

$$x^3 - 6x - 4 = (x + 2)(x^2 - 2x - 2).$$

PROBLEM SET 3–6

1. Perform the following divisions. Express the answer in both forms of Eq. (3–12). Divide:

(a) $x^3 - 6x - 4$ by $x - 2$

(b) $x^3 - 6x - 4$ by $x^2 + x - 2$

(c) $x^3 - 12x - 16$ by $x - 2$ and by $x + 2$

(d) $x^3 - 12x - 16$ by $x^2 - 2x + 8$

(e) $x^3 - 12x - 16$ by $2x - 3$

(f) $x^4 + 3x^3 + 7x^2 + 7x + 6$ by $x^2 + 2x + 3$

(g) $x^4 + 3x^3 + 7x^2 + 7x + 6$ by $x^2 - 2x + 3$

2. (a) Evaluate $F(x) = x^4 - 3x^3 + 7x^2 - 3x - 14$ at $x = 1$, $x = -1$, $x = 2$ by substitution.

(b) Divide $F(x)$ by $x - 1$, by $x + 1$ and by $x - 2$ and check results by the Remainder Theorem.

(c) Find the linear and quadratic factors of $F(x)$.

3. (a) Evaluate $F(x) = x^4 + 5x^3 + 2x^2 - 15x - 9$ at $x = 1, -1, 3, -3$.

(b) Find the quotient of $F(x)$ by $x + 3$, and evaluate this quotient at $x = -3$.

(c) Find the linear and quadratic factors of $F(x)$.

4. (a) If a and b are real numbers, distinguish between the complete quotient and partial quotient if a is divided by b. (b) If $F(x)$ and $D(x)$ are polynomials in the variable x, distinguish between the complete quotient and partial quotient if $F(x)$ is divided by $D(x)$.

5. If $H(x, y)$ and $D(x, y)$ are homogeneous polynomials in the variables (x, y) (for definition see Section 2-5), show how Eqs. (3-11) and (3-12) apply to these.

6. Divide

(a) $a^3 - b^3$ by $a - b$

(b) $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$

(c) $a^3 - 3a^2b + 3ab^2 - b^3$ by $a + b$ and check by means of the Remainder Theorem

(d) $x^3 + x^2y + xy^2 + y^3$ by $x + y$ and check as in part (c)

(e) $x^3 - x^2y + xy^2 - y^3$ by $x + y$ and check as in part (c)

3-11 Factoring polynomials. The Remainder and Factor Theorems can be used in searching for factors of the first degree for a given polynomial.

EXAMPLE 3-27. To factor $F(x) = x^2 - x - 6$ over the set of integers, seek factors of the form $x - r$, where r is an integral factor of -6. Hence r must be in the set $\{1, -1, 2, -2, 3, -3, 6, -6\}$. Evaluate $F(x)$ by substitution for each such r. It is found that $F(3) = 9 - 3 - 6 = 0$, so that $x - 3$ is one factor; the other is then found by inspection: $x^2 - x - 6 = (x - 3)(x + 2)$ for all x.

EXAMPLE 3-28. To find factors of the form $x - r$ for $F(x) = x^2 + 4x + 6$, where r is an integer, it is observed that $-r$ must be positive, so the only possibilities are $r = -1, -2, -3, -6$.

$$F(-1) = 3, \quad F(-2) = 2, \quad F(-3) = 3, \quad F(-6) = 18.$$

Hence $x^2 + 4x + 6$ cannot be factored over the set of integers. If we note $x^2 + 4x + 6 = x^2 + 4x + 4 + 2 = (x + 2)^2 + 2$ and observe that

$F(x)$ is positive for all real numbers, we conclude again that it cannot be factored over the set of all real numbers.

The procedure for a homogeneous polynomial in two variables is similar to that for a nonhomogeneous polynomial in one variable, since these can be reduced as illustrated below.

EXAMPLE 3-29.

$$H(x, y) = 15x^2 - 11xy - 14y^2 = 15y^2\left(\frac{x^2}{y^2} - \frac{11xy}{15y^2} - \frac{14}{15}\right).$$

Let $x/y = u$; then $H(x, y) = 15y^2(u^2 - \frac{11}{15}u - \frac{14}{15})$ and the problem of factoring $H(x, y)$ over the set of integers becomes that of factoring $F(u) = u^2 - \frac{11}{15}u - \frac{14}{15}$ over the set of rational fractions. Any factor must be of the form $u - (a/b)$, where a is an integral factor of 14 and b is an integral factor of 15. It is easy to verify that $F(-\frac{2}{3})$ and $F(\frac{7}{5})$ are both zero:

$$F\left(-\frac{2}{3}\right) = \frac{4}{9} + \frac{11}{15}\cdot\frac{2}{3} - \frac{14}{15} = \frac{20 + 22 - 42}{45} = 0.$$

Hence

$$F(u) = \left(u + \frac{2}{3}\right)\left(u - \frac{7}{5}\right) = \left(\frac{x}{y} + \frac{2}{3}\right)\left(\frac{x}{y} - \frac{7}{5}\right),$$

so that

$$H(x, y) = 15y^2\left(\frac{x}{y} + \frac{2}{3}\right)\left(\frac{x}{y} - \frac{7}{5}\right) = (3x + 2y)(5x - 7y),$$

where the factor $15y^2$ has been absorbed into the other factors.

EXAMPLE 3-30. To factor $H(x, y) = x^3 + y^3$ over the set of integers, consider $x = y$ or $x = -y$. $H(y, y) = y^3 + y^3 \neq 0$, so that $x - y$ is not a factor of $x^3 + y^3$. However, $H(-y, y) = -y^3 + y^3 = 0$, so that $x + y$ is such a factor and the residual factor is found by division (see Eq. 2-15):

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \quad \text{for any } x \text{ and any } y.$$

EXAMPLE 3-31. If $H(a, b) = a^4 - b^4$, then $H(a, a) = 0$ and $H(a, -a) = 0$, so that $a - b$ and $a + b$ are both factors of $a^4 - b^4$. If we use Eq. (2-10), involving the difference of two squares:

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2).$$

Since replacing b in $a^2 + b^2$ by either a or $-a$ does not give zero, $a^2 + b^2$ cannot be factored over the set of integers.

EXAMPLE 3-32. To factor $F(x) = x^4 + 5x^3 + 2x^2 - 15x - 9$ over the set of integers, seek factors of the form $(x - r)$, where r is an integral factor of -9, and $F(r) = 0$. It is found that $F(1)$, $F(-1)$, $F(3)$ are all different from 0.

$$F(-3) = 81 - 135 + 18 + 45 - 9 = 144 - 144 = 0.$$

Hence $x + 3$ is a factor of $F(x)$ and a second factor $Q(x)$ is found to be $Q(x) = x^3 + 2x^2 - 4x - 3$, so that

$$F(x) = (x + 3)Q(x), \quad \text{for any } x.$$

$$\begin{array}{r}
x^3 + 2x^2 - 4x - 3 \\
\hline
x + 3 \overline{)\, x^4 + 5x^3 + 2x^2 - 15x - 9} \\
\underline{x^4 + 3x^3} \\
2x^3 + 2x^2 \\
\underline{2x^3 + 6x^2} \\
-4x^2 - 15x \\
\underline{-4x^2 - 12x^2} \\
-3x - 9 \\
\underline{-3x - 9} \\
R = 0
\end{array}$$ Check.

Now $F(1) \neq 0$ and $F(1) = 4Q(1)$. Hence $Q(1) \neq 0$. Similarly $Q(-1) \neq 0$ and $Q(3) \neq 0$, so that the only possible factor of $Q(x)$ is $x + 3$. Also $Q(-3) = -27 + 18 + 12 - 3 = 0$, so that $F(x) = (x + 3)^2 Q_1(x)$, where the factor $Q_1(x) = x^2 - x - 1$ is found by dividing $Q(x)$ by $x + 3$. As in the argument given above, $Q_1(1) \neq 0$ and $Q_1(-1) \neq 0$, so that $Q_1(x)$ cannot be factored over the set of integers. Hence

$$F(x) = (x + 3)^2(x^2 - x - 1), \quad \text{for any } x.$$

To simplify algebraic fractions by dividing numerator and denominator by the same common factor, the first step is to factor both. This is illustrated by several examples.

EXAMPLE 3-33.

$$\frac{6x^2 - x - 1}{2x^2 + 5x - 3} = \frac{(2x - 1)(3x + 1)}{(2x - 1)(x + 3)} = \frac{3x + 1}{x + 3}.$$

EXAMPLE 3-34.

$$\frac{a^3 - b^3}{a^4 - b^4} = \frac{(a - b)(a^2 + ab + b^2)}{(a - b)(a + b)(a^2 + b^2)} = \frac{a^2 + ab + b^2}{(a + b)(a^2 + b^2)}.$$

It is observed that both numerator and denominator become 0 if $a = b$, indicating the presence of the common factor $a - b$. Neither $a^2 + b^2$ nor $a^2 + ab + b^2$ can be factored over the set of integers.

When adding algebraic fractions (see Section 3–5 and Problem Set 3–2), it was necessary to express each fraction as an equivalent fraction using a common denominator. It is not necessary to use the product of all the denominators, but the *least common denominator* may be used. This is the algebraic polynomial of lowest degree which is exactly divisible by all the denominators, and is best found when the denominators are first put in factor form. The least common denominator includes the highest power of any factor that occurs in any denominator. (Changes in sign are not considered as giving a different least common denominator.)

EXAMPLE 3–35. To add

$$\frac{1}{x^2 + x} - \frac{4}{x^2 - 1} + \frac{1}{x^2 - x}, \quad (x \neq 0, 1, -1),$$

note that the denominators are $x(x + 1)$, $(x - 1)(x + 1)$, and $x(x - 1)$, so that the least common denominator is $x(x - 1)(x + 1)$. Multiply the numerator and denominator of the first fraction by $x - 1$, those of the second fraction by x, and those of the third by $(x + 1)$. Then the sum is given by

$$\frac{(x - 1) - 4x + x + 1}{x(x - 1)(x + 1)} = \frac{-2x}{x(x - 1)(x + 1)} = \frac{-2}{(x - 1)(x + 1)}.$$

As a check, use $x = 2$. The original fractions have the value

$$\frac{1}{6} - \frac{4}{3} + \frac{1}{2} = \frac{1 - 8 + 3}{3} = \frac{-4}{6} = \frac{-2}{3},$$

while direct evaluation of the result gives the same fraction.

EXAMPLE 3–36. Consider the sum

$$S = \frac{1}{a^2 - 3ab + 2b^2} + \frac{1}{a^2 - b^2} + \frac{1}{a^2 + 3ab + 2b^2}, \quad (a \neq \pm b, \pm 2b).$$

$$S = \frac{1}{(a - b)(a - 2b)} + \frac{1}{(a - b)(a + b)} + \frac{1}{(a + b)(a + 2b)}.$$

The least common denominator is $(a - b)(a + b)(a - 2b)(a + 2b)$.

$$S = \frac{(a^2 + 3ab + 2b^2) + (a^2 - 4b^2) + (a^2 - 3ab + 2b^2)}{(a^2 - b^2)(a^2 - 4b^2)}$$

$$= \frac{3a^2}{(a^2 - b^2)(a^2 - 4b^2)}.$$

As a check, let $a = 3b$:

$$S = \frac{1}{2b^2} + \frac{1}{8b^2} + \frac{1}{20b^2} = \frac{1}{b^2}\left(\frac{20 + 5 + 2}{40}\right) = \frac{27}{40b^2};$$

also,

$$\frac{27b^2}{(8b^2)(5b^2)} = \frac{27}{40b^2}, \text{ affording a check.}$$

Problem Set 3–7

Use the Remainder and Factor Theorems to factor the polynomials in problems 1 to 22 into factors of the lowest degree over the set of integers, where possible. Otherwise show that such factorization is impossible.

1. $3x^2 - 11x + 6$
2. $9x^2 - 18x + 8$
3. $3x^2 + 6x - 8$
4. $15x^2 + 22x + 8$
5. $x^2 - 2x - 4$
6. $3x^2 - 12x + 8$
7. $x^3 + y^3$
8. $x^2 - xy + y^2$
9. $x^3 - 27y^3$
10. $16x^2 - 72xy + 81y^2$
11. $12a^2 - ab - 20b^2$
12. $12a^2 - 35ab + 25b^2$
13. $x^3 - 6x^2 + 11x - 6$
14. $x^3 - 6x^2 + 7x + 6$
15. $x^3 - 2x - 1$
16. $x^3 + 2x - 1$
17. $x^3 + 7x^2 + 16x + 12$
18. $x^3 - 9x^2 + 27x - 27$
19. $x^4 + 6x^3 + 12x^2 + 10x + 3$
20. $x^4 + 4x^3 - 2x^2 - 12x + 9$
21. $x^4 - 3x^3 + 2x^2 - 5x - 3$
22. $x^4 + 2x^3 - 6x^2 - 16x - 8$

Simplify the following fractions. Include a check.

23. $\dfrac{x^2 - 25}{2x^2 + 5x - 25}$
24. $\dfrac{2x^2 - 15x + 25}{2x^2 + 5x - 25}$
25. $\dfrac{4a^2 + 4ab - 3b^2}{2a^2 + 5ab + 3b^2}$
26. $\dfrac{a^3 + 2a^2b + 2ab^2 + b^3}{a^3 - b^3}$
27. $\dfrac{a^3 + b^3}{a^2 - b^2}$
28. $\dfrac{(1/x^3) - (1/y^3)}{(1/x^2) - (1/y^2)}$

Add the following sets of fractions and simplify the results. Include a check.

29. $\dfrac{4}{(x-1)^2} - \dfrac{4}{x^2 - 1}$
30. $\dfrac{4}{x-2} - \dfrac{8}{x^2 - 4}$
31. $\dfrac{1}{(x-2)^2} - \dfrac{2}{x^2 - 4} + \dfrac{1}{(x+2)^2}$
32. $\dfrac{1}{2x^2 + x - 1} - \dfrac{2}{2x^2 + 3x - 2} + \dfrac{1}{2x^2 + 5x - 3}$

CHAPTER 4

LINEAR EQUATIONS

4–1 Equations. Algebra employs declarative sentences, expressed wholly or partially in symbolic form, which relate two numbers or algebraic expressions. The common and highly important relationship of equality is stated in such sentences and when these sentences are expressed in symbolic form, they are called *equations*. Declarative sentences involving equations can be written which may not be true, and the symbols used may have specific meanings or may be restricted to some specified set of numbers. Sentences which contain variables (referred to as open sentences) may be true for some values of the variables and fail to be true for other values of these variables. In such cases it may be required to find the values of the variables for which the sentence is true. For example, the sentence "$3 + 4 = 7$" is true; the sentence "$7 - 3 = 5$" is not true; the sentence "$a + b = b + a$" is true for any a and any b; the sentence $a + x = b$ is true for one and only one value of x, namely, $x = b - a$; the sentence "$a + 2a = b$" is not true for any a and any b, but it is true for any a and b for which $b = 3a$.

A true sentence is often preceded by an appropriate label. The labels "Axiom," "Postulate," "Law," "Definition" all imply that the sentence is true by assumption, that is, it is taken as true without proof. The label "Theorem" (or one of its synonyms) indicates that the truth of the sentence follows by logical reasoning from the stated assumptions. Such sentences include legends stating the values of the variables for which the sentences are true.

Equations which are true for all values of the variables are called *identities*. Most of the fundamental laws of addition and multiplication are identities. Thus, the distributive law, $a(b + c) = ab + ac$, is an identity, since it is assumed true for any a, b, c. The special products of Section 2–6 are identities, being true for any values of the variables. Equations which are not true for all values of the variables are called *conditional equations*, and it is a fundamental problem to find the sets of numbers for which such equations are true. Such sets are called the *solution sets* for the conditional equations, and each member of the solution set is called a *solution*. For example, it has been proved (Section 1–5) that the equation $x + a = b$ is true for one and only one value of x: $x = b - a$. It was assumed (Section 3–1) that the equation $ax = 1$ had one and only one solution for any a different from zero, namely, $x = 1/a$. It has been proved in Section 2–2 that the equation $x(x - 2) = 0$ has two and only

two solutions, namely $x = 0$ and $x = 2$. A conditional equation might have no solution in the specified domains of the variables, in which case the solution set is the empty set. For example, if x is restricted to be an integer, then the equation $2x = 3$ has no solution; if x is restricted to be a real number, then $(x - 2)^2 = -3$ has no solution. In general, it is not known in advance whether or not an equation has solutions in a specified domain. It is always important to know the set of numbers to which the variable is restricted, for one may formally find solutions of the equation which are not in this restricted set. In many applied problems, the variables are restricted to the positive numbers.

This section treats equations in one variable. If other symbols appear in these equations, they represent fixed real numbers. Unless indicated otherwise, the domain of the variable is understood to be the set of real numbers. Two equations are said to be *equivalent* if they have the same solution set. Starting with a given equation, certain arithmetical operations may be performed upon this equation to give new equations that are equivalent to the original one. The process of solving an equation involves finding a sequence of equivalent equations, the last of which has readily recognized solutions. An operation on an equation which leads to an equivalent equation is said to be *permissible*. The following operations are permissible:

(1) *Removal or insertion of parentheses.*

This includes the expansion of any indicated product and factoring. It is justified because, for any x, these operations merely change the form of the numbers.

(2) *Addition of the same algebraic expression to both members of the equation.*

(Subtraction is included as addition of the additive inverse.)

This operation is permissible, because if for any x in the specified domain, the members of the equation have the values a and b, and the algebraic expression has the value c, then

$$a = b \quad \text{if and only if} \quad a + c = b + c.$$

(3) *Multiplication of both members of the equation by the same algebraic expression which cannot be zero for any values of x in the specified domain.*

(Division is considered as multiplication by the multiplicative inverse.)

This operation is permissible, because if for any x in the specified domain, the members of the equation have the values a and b, and the algebraic expression has the value $c \neq 0$, then

$$a = b \quad \text{if and only if} \quad ac = bc.$$

The limitation $c \neq 0$ is essential not only so that the appropriate laws

may be applied but also to maintain equivalence. The equation $0 \cdot a = 0 \cdot b$ is true for any x, even though $a = b$ may not be true for any x.

The concept of equivalent equations is more than an application of the laws "equals added to equals give equals" and "equals multiplied by equals give equals," since it involves the more basic idea that the equations have the same solution set.

EXAMPLE 4–1. The equation $x = 2$ is an equation with one and only one solution, whereas $x^2 = 4$ is also an equation which is true for $x = 2$. The equation $x^2 + x = 4 + 2$ is also true when $x = 2$. However, the original equation and this final equation are not equivalent, for $x^2 + x = 6$ is also true for $x = -3$, whereas the equation $x = 2$ is not true for $x = -3$. It is noted that the same algebraic expression was not added to both members of the original equation in order to obtain the final equation. The equation $x = 2$ is, however, equivalent to the equation $x + 4 = 2 + 4$ and to the equation $x^2 + x = x^2 + 2$.

EXAMPLE 4–2. Multiplication by an algebraic expression which involves the variable may introduce extraneous solutions, and division by such an expression may lose some of the solutions. The equations $x - 2 = 0$ and $x(x - 2) = 0$ are not equivalent, for the first has only the solution $x = 2$ and the second has the solutions $x = 2$ and $x = 0$. The multiplication of both members of the first equation by x introduces an extraneous solution. If the second equation is solved by dividing both members by x, a solution is lost. Such division is not permissible, since it would involve division by zero.

EXAMPLE 4–3. Find all the real solutions of the equation

$$x^3 - 8x + 2 = x^2 - 4.$$

First verify that $x = 3$ is one solution.

If $x = 3$, the left member takes the form $27 - 24 + 2$, and the right member takes the form $9 - 4$; these are different forms of the number 5. Subtract $x^2 - 4$ from both members of the equation (Operation 2) to obtain the equivalent equation

$$x^3 - x^2 - 8x + 6 = 0.$$

Since $x = 3$ is one solution, $(x - 3)$ is one factor of the polynomial on the left, so this equation is equivalent (Operation) 1 to

$$(x - 3)(x^2 + 2x - 2) = 0,$$

where the second factor was found by division (Section 3–10). Inasmuch

as a product is zero when either factor is zero, the solutions of this last equation are $x = 3$ together with the solutions of the equation

$$x^2 + 2x - 2 = 0.$$

Solutions of $x^2 + 2x - 2 = 0$ give the remaining real solutions of the original equation.

4–2 Linear equations in one variable. The equation

$$ax + b = 0, \quad (a \neq 0), \tag{4–1}$$

where a and b are real numbers and x is a variable, is called a *linear equation* in x. It has the unique solution $x = -b/a$. The equations

$$ax + b = 0, \quad ax = -b, \quad x = -\frac{b}{a}, \quad (a \neq 0),$$

obtained through permissible operations are equivalent, and the last one gives the unique solution.

More generally, a *linear equation* in the variable x is an equation which can be reduced to the form of Eq. (4–1) by means of permissible operations and hence has one and only one solution.

EXAMPLE 4–4. Examples of such equations are

(a) $ax + b = cx + d$, a, b, c, d real numbers, $a \neq c$;

(b) $(x - a)^2 = (x - b)^2$, a, b real numbers, $a \neq b$;

(c) $\dfrac{ax}{x - b} = c$, $x \neq b$, a, b, c real numbers, $a \neq c$.

To reduce example (a) to the form of Eq. (4–1), first subtract b and then cx from both members of the equation. It becomes, after inserting parentheses,

$$(a - c)x = d - b, \quad (a - c) \neq 0.$$

The permissible operation of subtracting a term of one member of the equation from both members is equivalent to the operation of changing a term from one member of the equation to the other and reversing its sign. This is called a *transposition*. From $ax + b = cx + d$, we obtain by transposition the equivalent equation

$$ax - cx = d - b \quad \text{or} \quad (a - c)x = d - b.$$

If $a \neq c$, the solution is $x = (d - b)/(a - c)$. If $a = c$, the equation would have no solution unless $d = b$. If $a = c$ and $d = b$, any x would satisfy the equation.

Example 4–4(b) becomes successively, by means of permissible operations,

$$x^2 - 2ax + a^2 = x^2 - 2bx + b^2$$
$$-2ax + a^2 = -2bx + b^2$$
$$(2b - 2a)x = b^2 - a^2$$
$$x = \frac{b^2 - a^2}{2(b - a)} = \frac{a + b}{2}, \qquad (a \neq b).$$

If a were equal to b, the original equation would be an identity, since it would be true for any value of x.

In Example 4–4(c), the equation $ax/(x - b) = c$ is not defined if $x = b$. For $x \neq b$, multiply both members by $x - b$ to obtain the equivalent equation $ax = cx - bc$, which is a linear equation as long as $a \neq c$, and its solution is $x = bc/(c - a)$, $c \neq a$. If $a = c \neq 0$, then $ax = ax - ab$ would have no solution unless $b = 0$, in which case every x satisfies the original equation.

Linear equations involving algebraic fractions often occur as formulas in the physical or social sciences. These formulas may contain several symbols, all except one of which are constants while this one is a variable. For example:

(a) $C = 2\pi r$ may be considered as a linear equation in r;
(b) $1/R = 1/R_1 + 1/R_2$ may be considered as a linear equation in R;
(c) $P = (A)/(1 + ni)$ may be considered as a linear equation in i;
(d) $S = (a - rl)/(1 - r)$ may be considered as a linear equation in r.

Example 4–5. The foregoing formulas are solved as follows:

(a) $C = 2\pi r, \qquad r = \frac{C}{2\pi}$;

(b) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \qquad \frac{1}{R} = \frac{R_1 + R_2}{R_1 R_2}, \qquad R = \frac{R_1 R_2}{R_1 + R_2}$;

(c) $P = \frac{A}{1 + ni}, \qquad 1 + ni = \frac{A}{P}; \qquad ni = \frac{A}{P} - 1 = \frac{A - P}{P},$
$i = \frac{A - P}{nP}$;

(d) $S = \frac{a - rl}{1 - r}, \qquad S - rS = a - rl, \qquad r(l - S) = a - S,$
$r = \frac{a - S}{l - S}$.

It is assumed in every case that the denominators are not zero.

Problem Set 4–1

1. (a) Are the equations $x = 3$, $x^2 = 9$ equivalent? Explain.
   (b) If x is restricted to be real, are the equations $x = 3$, $x^3 = 27$ equivalent? Explain.
   (c) Are the equations $x = 3$, $x^2 + x = x^2 + 3$ equivalent? Explain.
   (d) If $x = 3$, then $x^2 = 9$, so that $x^2 + x = 12$. Is this last equation equivalent to either of the other two?

2. (a) Are the equations $x^2 - 4x = 0$ and $x = 4$ equivalent?
   (b) Are the equations $x/(x - 2) = 3$ and $x = 3(x - 2)$ equivalent?
   (c) If the equation $[x/(x - 2)] + [x/(x + 2)] = 2$ is multiplied by $(x - 2)(x + 2)$, a new equation is obtained. Are the two equations equivalent? Explain. What can be said about the solutions of the given equation?

3. Solve the following linear equations.
   (a) $5x + 12 = -4x + 20$
   (b) $-9y + 11 = 6y - 15$
   (c) $-9z - 11 = -6z - 15$
   (d) $3x + 4[2x - 3(x + 2)] = 24 - 6x$
   (e) $12y - [4 - 2(y - 2) - 16] - (4 - y) = 0$

4. Solve the following linear equations.
   (a) $(x - 4)^2 = x^2 - 4$
   (b) $(x - 4)^2 = (x - 2)^2$
   (c) $(x - 4)^2 = (x + 4)^2$
   (d) $(2x + 3)^2 + (3x + 2)^2 = 13x^2 + 49$

5. Solve the following linear equations.
   (a) $\dfrac{4x}{x - 2} = 12$ (b) $\dfrac{4x}{x - 2} = 4$
   (c) $\dfrac{2x}{x - 4} = 2$ (d) $\dfrac{2x}{x + 4} = 4$
   (e) $\dfrac{2x - 4}{x + 2} = 1$ (f) $\dfrac{2x - 4}{x + 2} = 2$

6. In each of the following formulas, solve for the indicated symbol.
   (a) $\dfrac{1}{x} = \dfrac{1}{y} + \dfrac{1}{z}$ Solve for x and also for y.
   (b) $V = \frac{1}{3}\pi r^2 h$ Solve for h.
   (c) $P = \dfrac{A}{1 + ni}$ Solve for n.
   (d) $A = P(1 - nd)$ Solve for d.

(e) $\frac{x}{a} + \frac{y}{b} = 1$ Solve for y.

(f) $p = 40 - 5x$ Solve for x.

(g) $p = \frac{8}{4 + x}$ Solve for x.

(h) $Ax + By + C = 0$ Solve for y and also for x.

(i) $C = \frac{5}{9}(F - 32)$ Solve for F.

4–3 Worded problems and linear equations. Problems related to such concepts as length, area, volume, uniform motion, and levers can often be reduced to the solution of linear equations. To solve such a problem, let x represent one of the unknowns. Using the data given in the problem, write an equation that relates the other quantities to x. Solve this equation for x. If this equation is linear, it can be solved by means of the permissible operations. Although no general procedure can be given for obtaining the required equation, methods used are illustrated by a number of examples.

(1) *Sum and difference of numbers.*

EXAMPLE 4–6. The sum of two numbers is 24; one number is three more than twice the other. Find the numbers.

Let x be the smaller number; then the other number is $2x + 3$. Since their sum is 24,

$$x + (2x + 3) = 24$$
$$3x = 21$$
$$x = 7, \quad 2x + 3 = 17 \quad \text{Ans.}$$

EXAMPLE 4–7. Can the sum of three consecutive odd integers be (a) 25? (b) 45?

(a) Let x be the smallest integer, so that the others are $x + 2$ and $x + 4$. Then

$$x + (x + 2) + (x + 4) = 25$$
$$3x + 6 = 25$$
$$3x = 19$$
$$x = \tfrac{19}{3}.$$

Since $\frac{19}{3}$ is not an integer, there are no such odd integers.

(b) If 25 is replaced by 45, the equation takes the form $3x = 39$, or $x = 13$. The three consecutive odd integers are thus 13, 15, 17.

In general, if the sum of three consecutive odd integers is to be the number N, then N must be an integral multiple of 3.

EXAMPLE 4–8. The sum of two numbers is 25 and the difference of their squares is 225. Find the numbers.

Let x be the larger of the numbers; the other is then $25 - x$. The second statement in the problem yields

$$x^2 - (25 - x)^2 = 225.$$

Then $$x^2 - (625 - 50x + x^2) = 225$$

$$50x = 225 + 625 = 850$$

$$x = 17, \quad 25 - x = 8 \quad \text{Ans.}$$

(2) *Division of a line segment.*

EXAMPLE 4–9. A line segment AB is $7\frac{1}{2}$ in. long. Locate the point C between A and B so that AC is $\frac{3}{2}$ in. shorter than twice CB.

Let x = length of AC in inches. Then $7\frac{1}{2} - x$ is the length of CB in inches. The given statement yields

$$x = 2(7\tfrac{1}{2} - x) - \tfrac{3}{2}$$

$$3x = 15 - \tfrac{3}{2} = \tfrac{27}{2}$$

$$x = \tfrac{9}{2} = AC; \qquad CB = 3. \quad \text{Ans.}$$

EXAMPLE 4–10. On a graduated ruler find the points which trisect the part of the scale between 2 and $4\frac{5}{8}$.

Call the points (Fig. 4–1) of trisection C and D, where $AC = \frac{1}{3}AB$ or $AC = \frac{1}{2}CB$, and $AD = \frac{2}{3}AB$. Let the scale readings for C and D be x and y, respectively. Then

$$x - 2 = \tfrac{1}{3}(4\tfrac{5}{8} - 2) \qquad \text{or} \qquad x - 2 = \tfrac{1}{2}(4\tfrac{5}{8} - x)$$

The first of these equations yields

$$x = 2 + \tfrac{1}{3} \cdot \tfrac{21}{8} = 2\tfrac{7}{8}.$$

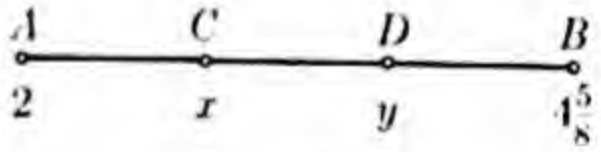


FIGURE 4–1

We leave it to the reader to verify that the second equation gives the same result. Also:

$$y - 2 = \tfrac{2}{3} \cdot \tfrac{21}{8} \qquad \text{or} \qquad y = \tfrac{15}{4}.$$

(3) *Perimeter and area.*

EXAMPLE 4–11. A rectangle 4 in. by 8 in. is completely bordered by a strip x in. wide. If the perimeter of the larger rectangle is twice that of the smaller rectangle, what is the value of x?

The perimeter of the original rectangle is $(8 + 4 + 8 + 4)$ in. and that of the bordered rectangle is $(8 + 2x) + (4 + 2x) + (8 + 2x) + (4 + 2x)$. Hence

$$\begin{aligned} 24 + 8x &= 2 \cdot 24 \\ 8x &= 24 \\ x &= 3. \end{aligned}$$

EXAMPLE 4–12. A rectangle is twice as long as it is wide. If it is bordered by a strip 2 ft wide, its area is increased by 160 sq. ft. What are its dimensions?

Let its width in feet be x. Then its length is $2x$. The dimensions of the bordered rectangle in feet are $x + 4$ and $2x + 4$. Hence

$$\begin{aligned} (x + 4)(2x + 4) &= 2x^2 + 160 \\ 2x^2 + 12x + 16 &= 2x^2 + 160 \\ 12x &= 144 \\ x = 12, \quad 2x &= 24. \quad \text{Ans.} \end{aligned}$$

(4) *Division of estates or investments.*

EXAMPLE 4–13. An estate valued at \$6000 is to be divided among a wife, a son, and a daughter. Each is first to receive \$500 and then the rest of the estate is to be divided so that the wife receives three times as much as the daughter and the son receives twice as much as the daughter. How much does each receive?

After the \$1500 has been deducted, let x be the amount of the estate, expressed in dollars, received by the daughter. The amounts received by the three heirs are $500 + x$, $500 + 2x$, $500 + 3x$. Then

$$\begin{aligned} (500 + x) + (500 + 2x) + (500 + 3x) &= 6000 \\ 6x &= 4500 \\ x &= 750. \end{aligned}$$

Hence the amounts received are \$1250, \$2000, \$2750.

EXAMPLE 4–14. An investor with \$40,000 wants to receive an over-all annual return of approximately 5%. He invests his funds in government bonds giving an annual return of $4\frac{1}{4}\%$ and in stocks giving a probable annual return of 6%. He desires to keep his stock investment to a minimum in order to lessen his risk. The investments are made in units of \$100 each. How many bonds and how many stocks should he purchase?

Let x = number of bonds, each paying \$4.25 a year. Then $400 - x$ = number of stocks, each paying \$6.00 a year. His desired income in dollars

is 5% of 40,000 = 2000, and his probable income in dollars is $\$4.25x + 6(400 - x)$, so that

$$2000 = (4.25 - 6)x + 2400$$
$$\tfrac{7}{4}x = 400$$
$$x = \tfrac{1600}{7},$$

and to the nearest integer $x = 229$. Hence he should buy 229 bonds and 171 stocks each at \$100. His actual annual income would be

$$229(\$4\tfrac{1}{4}) + 171(\$6) = \$1999.25$$

and his over-all annual return would be approximately 5%.

(5) *Mixtures.*

EXAMPLE 4–15. How much alcohol must be added to 50 gallons of a mixture of water and alcohol which is 85% alcohol to make a mixture which is 90% alcohol?

Let y = number of gal of alcohol to be added, so that the new mixture contains $50 + y$ gal. The number of gal of alcohol after the addition is $(85\% \cdot 50) + y$ and also $90\%(y + 50)$. Hence by equating these two values and multiplying by 100, we obtain

$$4250 + 100y = 90y + 4500$$
$$10y = 250$$
$$y = 25.$$

EXAMPLE 4–16. A grocer mixes coffee which sells for \$.80 per lb with coffee which sells for \$.50 per lb to make 100 pounds of coffee that he can fairly sell for \$.75 a pound. How many pounds of each grade does he use?

Let x = the number of pounds of \$.75 coffee
Then $100 - x$ = the number of pounds of \$.50 coffee.

If we equate the values of the coffee (in cents), we have

$$80x + 50(100 - x) = 7500$$
$$30x = 2500$$
$$x = 83\tfrac{1}{3}, \quad 100 - x = 16\tfrac{2}{3}. \quad \text{Ans.}$$

(6) *Levers.*

When a horizontal bar is supported at a point P about which it is free to rotate and a force F acts at a distance d (called the lever arm) from P (called the fulcrum), the moment of force about P is the product Fd.

When a number of forces act at different distances from P, the lever is in equilibrium if the sum of the moments of forces which tend to produce counterclockwise rotation about P equals the sum of the moments of force which tend to produce clockwise rotation about P. If the lever is uniform, the weight of the lever may be considered as concentrated at its center.

EXAMPLE 4–17. Two boys weighing 75 and 55 lb, respectively, sit on opposite ends of a 10-ft teeter board. The board weighs 20 lb. Where is the point of support if the board is just balanced?

Let x (Fig. 4–2) be the distance in feet from the fulcrum P to the center C. Then the lever arm for the 75-lb boy is $5 - x$, and that for the 55-lb boy is $5 + x$. Hence

$$55(5 + x) + 20x = 75(5 - x)$$
$$275 + 75x = 375 - 75x$$
$$150x = 100$$
$$x = \tfrac{2}{3}$$

Hence the point of support is $\frac{2}{3}$ ft or 8 in. from the center of the board.

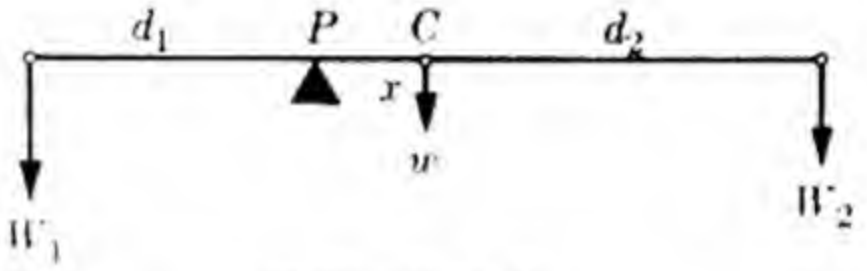


FIGURE 4–2

(7) *Uniform motion.*

When an object moves with a constant speed r for time t, the distance it travels is $d = rt$, where d, r, t are each expressed in appropriate units.

EXAMPLE 4–18. Two cars are traveling at speeds of 45 mph and 60 mph. The second car is one mile behind the first. How long will it take the second car to overtake the first car?

Let t = number of hours. The first car travels $45t$ mi and the second travels $60t$ mi.

Therefore

$$45t + 1 = 60t$$
$$15t = 1, \quad t = \tfrac{1}{15}.$$

The time required for the second car to overtake the first car is $\frac{1}{15}$ hr or 4 min.

The result could be obtained by recognizing that the relative speed of the two cars is $(60 - 45)$ mph, and hence to gain 1 mi takes $\frac{1}{15}$ of an hour.

EXAMPLE 4–19. Two towns, A and B, are 250 mi apart. A car travels from A to B at the rate of 40 mph. Two hours after the first car starts, a second car leaves B for A and travels at the rate of 55 mph. When and where do they meet?

Let t be the number of hours the first car travels. The second car travels for $(t - 2)$ hours. In terms of distance, we have

$$\begin{aligned} 40t + 55(t - 2) &= 250 \\ 95t &= 360 \\ t &= \tfrac{360}{95} = 3\tfrac{15}{19}. \end{aligned}$$

The first car travels $3\frac{15}{19}$ hr and hence $\frac{72}{19} \cdot 40 = 151.6$ mi. The second car travels $1\frac{15}{19}$ hr and hence $\frac{34}{19} \cdot 55 = 98.4$ mi. The sum of these distances is 250 mi, which checks the results.

PROBLEM SET 4–2

1. The sum of two numbers is 18; their difference is 4. Find the numbers.

2. The sum of two numbers is 18; one number is 3 less than twice the other. Find the numbers.

3. (a) Find four consecutive integers whose sum is 42. (b) What restriction must be placed on the integer A so that the sum of four consecutive integers is A.

4. (a) Find three consecutive odd integers whose sum is the integer A. What restriction must be placed on A? (b) What is the smallest value greater than 100 that A may have?

5. The sum of two numbers is 10; the difference of their squares is 25. Find the numbers.

6. (a) The difference of the squares of two consecutive odd integers is the integer A. Show that A must be divisible by 8. (b) If $A = 40$, find the numbers. (c) What is the smallest value greater than 100 that A can have? Find the corresponding numbers.

7. A line segment AB is 10 cm long. Locate the point C between A and B (a) so that AC is $1\frac{3}{4}$ cm shorter than CB; (b) so that AC is $1\frac{1}{2}$ cm longer than $\frac{1}{3}$ of CB.

8. On a graduated ruler, where are the points that (a) bisect and (b) trisect the part of the scale between $2\frac{1}{2}$ and $6\frac{1}{4}$?

9. A line segment AB is 9 in. long. Locate the point C on this segment (a) so that one-half of AC equals one-third AB plus one-half of CB; (b) so that one-half of AC equals two-thirds of AB plus one-half of CB. Explain the answer in part (b).

10. A piece of wire 18 in. long is bent into a rectangle. Find the dimensions of the rectangle if (a) one side is 2 in. longer than the other; (b) one side is $\frac{3}{2}$ in. longer than twice the other.

11. A rectangle 6 in. by 10 in. is completely bordered by a strip x in. wide. If the perimeter of the larger rectangle is $\frac{3}{2}$ that of the smaller rectangle, what is the value of x?

12. A rectangle is three times as long as it is wide. If it is bordered by a strip 1 ft wide, its area is increased by 28 sq. ft. What are its original dimensions?

13. An estate valued at \$12,000 was to be divided among three heirs as follows: each one to obtain \$2000, and the residual to be divided among them in shares in the ratio of $2:2\frac{1}{2}:3\frac{1}{2}$. How was the estate divided?

14. An estate value at \$20,000 was to be divided among the widow, a son, and two daughters. The daughters were to receive equal shares, the son was to receive \$1000 more than each daughter, and the wife was to receive \$1000 more than twice that received by the son. How much did each receive?

15. An investor with \$50,000 wants to receive an annual income of \$3000. He can invest his funds in 4% government bonds and with a greater risk, in 7% mortgage bonds, in units of \$100 each. How many of each should he purchase if he desires to minimize his risk yet earn \$3000?

16. A trust fund of \$25,000 was established for John Jones. Equal amounts were invested in 4% government bonds and $5\frac{1}{2}$% preferred stocks. The rest was invested in 9% mortgages. The annual income was \$1500. How much was invested in each?

17. How much water must be added to 50 gal of a mixture of water and alcohol which is 85% alcohol to make a mixture which is 70% alcohol?

18. A dealer combines two mixtures of water and alcohol, one containing 45% alcohol, the other 70% alcohol, to make 5 gal of a mixture containing 55% alcohol. How much of each does he use?

19. A jobber mixes two grades of candy, one worth \$.85 per pound, the other worth \$.60 per pound, to make 50 lb of candy to sell at \$.75 per pound. How many pounds of each does he use?

20. How much cream containing 15% butter fat may be removed from 100 lb of raw milk containing 4% butter fat so that the remaining mixture contains 3% butter fat?

21. A and B weigh 150 and 180 lb, respectively. They sit on opposite ends of a 12-ft seesaw. Where must the fulcrum be placed in order that they balance each other (a) provided the weight of the board is neglected; (b) provided the board weighs 30 lb.

22. A, B, and C weigh 60, 80, and 120 lb respectively. A and C sit on opposite ends of a 14-ft teeter board whose fulcrum is at its center. Where should B sit in order that the board should just balance?

23. A, B, and C weigh 60, 80, and 120 lb, respectively. A and C sit on opposite ends of a 14-ft teeter board which weighs 40 lb, and B sits 4 ft in front of A. Where is the fulcrum placed if the board is to be balanced?

24. A, B, and C weigh 60, 80, and 100 lb, respectively. A and C sit on opposite ends of a 14-ft teeter board which weighs 40 lb, and B sits 1 ft in front of A. Where is the fulcrum placed if the board is to be balanced? Give a physical interpretation of any negative answer obtained.

25. Two cars are traveling at rates of 50 and 55 mph. The second car is 4 mi behind the first. How long will it take to overtake the first car?

26. Two towns are 200 mi apart. A car travels from the first toward the second at the rate of 35 mph. Three hours later a car leaves the second town for the first going 60 mph. When and where do they meet?

27. A freight train leaves Chicago for Denver and travels at an average speed of 35 mph. Five hours later a passenger train leaves Chicago for Denver and travels at an average speed of 60 mph. When and where will the passenger train overtake the freight?

28. The hands of a clock are together at noon. When will they be together again for the first time?

29. *A* and *B* are running around a circular 440-yd track at rates of 500 yd per min and 450 yd per min. (a) If they start at the same place and go in the same direction, when and where will they first be together again? (b) If they start at the same place and go in opposite directions, when and where will they first pass each other?

30. A man walked 15 mi and returned immediately by car which traveled at the rate of 40 mph. The entire trip required 5 hr. What was his average rate of walking?

31. How long would it take a train 200 ft long and traveling 60 mph to pass (a) a signal? (b) a station platform 320 ft long? (c) a second train 240 ft long going in the same direction at a speed of 40 mph? (d) a second train 240 ft long going in the opposite direction at a speed of 40 mph?

32. The total cost of an article was $24 and it was marked up to the selling price S. Later it was offered for sale at 15% off this selling price. The dealer still made a profit of 10% of his original cost. What was the marked-up price S?

4–4 Equivalent rates. Successive discounts. Section 3–8 discusses the percentage formula $P = rB$ and its relation to trade discount and net cost. This is now used to derive the formula for net cost under the conditions of successive trade discounts. Let L represent the list price, r the discount rate, D the discount, and C the net cost which is obtained by subtracting the discount from the list price. Then, for the case of a single discount,

$$D = rL, \quad C = L - rL = L(1 - r).$$

Suppose successive discount rates, r_1 and r_2, are given. Then on the basis of the first discount, the cost would be $C_1 = L(1 - r_1)$, and on the basis of second discount $C = C_1(1 - r_2)$, or

$$C = L(1 - r_1)(1 - r_2).$$

If r stands for the equivalent single discount rate, then $C = L(1 - r)$, so that

$$L(1 - r) = L(1 - r_1)(1 - r_2) = L[1 - (r_1 + r_2) + r_1r_2].$$

If this equation is solved for r:

$$r = r_1 + r_2 - r_1r_2. \tag{4–2}$$

For three successive discounts rates, r_1, r_2, r_3:

$$\begin{aligned} L(1 - r) &= L(1 - r_1)(1 - r_2)(1 - r_3) \\ &= L[1 - (r_1 + r_2 + r_3) + (r_1r_2 + r_2r_3 + r_3r_1) - r_1r_2r_3], \end{aligned}$$

so that

$$r = r_1 + r_2 + r_3 - r_1r_2 - r_2r_3 - r_3r_1 + r_1r_2r_3. \qquad (4\text{–}3)$$

This suggests the following theorem:

The discount rate equivalent to a set of successive trade discount rates is independent of the order of the discount rates.

A proof of this theorem for the case of two or three discount rates follows immediately from Eqs. (4–2) and (4–3). A proof for four discounts can be given in a similar way, while the general case can be treated through the use of mathematical induction, discussed in Chapter 10.

EXAMPLE 4–20. What single discount rate is equivalent to successive trade discount rates of 40%, 10%, and 5%?

Use Eq. (4–3), expressing the given rates as decimal fractions, to get

$$\begin{aligned} r &= 0.40 + 0.10 + 0.05 - [(0.40)(0.10) + (0.40)(0.05) + (0.10)(0.05)] \\ &\quad + (0.40)(0.10)(0.05) \\ &= 0.55 - 0.065 + 0.002 = 0.487 = 48.7\%. \end{aligned}$$

Graduated income tax. A recent federal income tax schedule for a joint return based upon the taxable income is 20% of the income between 0 and \$2000; 21% of the income between \$2000 and \$4000; 24% of the income between \$4000 and \$6000; 26% of income between \$6000 and \$8000, and 30% of that part of the income between \$8000 and \$10,000. A formula for the equivalent single rate for taxable incomes between \$6000 and \$8000 is derived as follows: Let the taxable income be B, the tax be T, and the equivalent single tax rate be r; then

$$\begin{aligned} T &= \$2000[20\% + 21\% + 24\%] + (B - \$6000)26\% \\ &= 26\%B - \$260. \end{aligned}$$

The tax T is also given by $T = rB$. Hence

$$rB = 26\%B - 260$$

$$r = 26\% - \frac{260}{B} = \left(26 - \frac{26000}{B}\right)\%. \qquad (4\text{–}4)$$

Note that the rate r depends upon the value of B, $(6000 \leqq B \leqq 8000)$.

EXAMPLE 4–21. Find the equivalent single tax rate, under the conditions stated above, for taxable incomes of \$6000, \$7500, and \$8000.

$$\text{If } B = \$6000, \qquad r = \left(26 - \frac{26000}{6000}\right)\% = (26 - 4\tfrac{1}{3})\% = 21\tfrac{2}{3}\%.$$

$$\text{If } B = \$7500, \qquad r = \left(26 - \frac{26000}{7500}\right)\% = (26 - 3.47)\% = 22.53\%.$$

$$\text{If } B = \$8000, \qquad r = \left(26 - \frac{26000}{8000}\right)\% = (26 - 3\tfrac{1}{4})\% = 22\tfrac{3}{4}\%.$$

Average speed. If an object moves with changing speeds, the average speed is defined by the equation $d_T = rt_T$, where d_T is the total distance traveled and t_T is the total time.

EXAMPLE 4–22. In traveling between two towns 150 mi apart, a freight train travels at the rate of 30 mph for the first 50 mi; it is then sidetracked for an hour and continues its journey at the rate of 50 mph. What is the average speed?

The total elapsed time, based on the formula $d = rt$, is

$$t_T = \tfrac{50}{30} + 1 + \tfrac{100}{50} = 4\tfrac{2}{3} \text{ (hr)}.$$

Hence the average speed is $150 \div 4\tfrac{2}{3} = 32.14$ (mph).

PROBLEM SET 4–3

1. What single trade discount rate is equivalent to the following set of successive trade discounts?

(a) 40%, 10%
(b) 30%, 15%
(c) 30%, 20%, 10%
(d) 40%, 10%, 10%

2. Using Eq. (4–4), find the equivalent single tax rate for a taxable income of \$6500 and \$7000. The results should be consistent with those of Example 4–21.

3. Using the data for income taxes given in Section 4–4, derive the formula for the equivalent single tax rate for a taxable income between \$4000 and \$6000. Check the formula using incomes of \$4000 and of \$6000. Find the equivalent rate for an income of \$5000.

4. Using the data for income taxes given in Section 4–4, derive the formula for the equivalent single tax rate for a taxable income between \$8000 and \$10,000. Check the formula using incomes of \$8000 and \$10,000. Find the equivalent rate for an income of \$9000.

5. In traveling between two stations 100 mi apart, a freight train travels at the rate of 25 mph for the first 40 mi; it is then delayed for half an hour and continues its journey at the rate of 40 mph. What is its average speed?

6. A student drives 45 mi to school and allows $1\frac{1}{2}$ hr for the trip. (a) One day he made the first 30 mi at the average rate of 40 mph. How fast must he drive for the remaining time to arrive early? (b) Another day he starts 25 min late, and travels the first 40 mi at the average rate of 35 mph. How fast must he drive the rest of the trip to arrive at the usual time? (c) On still another day he drives the first third of the distance at half the average rate and the remaining two-thirds of the distance at twice the average rate. Is he early, late, or on time?

4–5 Simple interest and simple discount. *Direct proportion.* If two variables X and Y are related so that $Y = kX$, k a given constant, we say that Y is *directly proportional* to X. Thus in the formula for uniform motion, $d = rt$, the distance is directly proportional to the time. In general, if (X_1, Y_1) and (X_2, Y_2) are any pairs of corresponding values, so that $Y_1 = kX_1$ and $Y_2 = kX_2$, then

$$\frac{Y_1}{X_1} = \frac{Y_2}{X_2} = k,$$

and the equality between the two fractions is called a *proportion.*

Simple interest. Interest, represented by the symbol I, denotes money paid at the end of the period of the loan for the use of other money, or for money earned by loaning other money. Thus, if money is borrowed, and if at the end of the loan period it is repaid together with an additional sum of money, this additional sum is the interest. The amount that is periodically paid on saving accounts, or earned by stocks and bonds, is interest.* The original sum of money is called the principal, represented by P; the time, represented by t is usually expressed in years. The interest is called *simple interest* provided it is directly proportional to the product of the principal and the time:

$$I = rPt, \tag{4–5}$$

where r, the constant of proportionality, is the interest rate. This is usually expressed as a decimal fraction per year or $(100r)\%$ per year. However, other time intervals, such as "interest at $1\frac{1}{2}\%$ per month," may be used.

The principal plus the interest is the amount, represented by A, to be repaid. P is also called the *present value* and A is called the *accumulated value.*

$$A = P + Prt = P(1 + rt). \tag{4–6}$$

Equation (4–6) is readily solved for P or r in terms of the other symbols:

$$P = \frac{A}{1 + rt}, \qquad r = \frac{A - P}{Pt} = \frac{I}{Pt}. \tag{4–7}$$

If three of the quantities P, A, r, t are given, the other can be computed.

* This is sometimes referred to as dividend.

Ordinary and exact interest. There is no single procedure used to measure the time t. If the time is stated in months, a year contains 12 months, each month is assigned 30 days, and fractions of a month are given in terms of the number of days. When the time is computed in this way, the interest is called *ordinary interest.* Ordinary interest, I_O, is given by

$$I_O = rP\frac{\text{no. of days}}{360},$$

where the no. of days = 30 (no. of months) + no. of extra days.

If the exact number of days is counted, and a year is considered to contain 365 days, the interest is called *exact interest.* Exact interest, I_E, is given by

$$I_E = rP\frac{\text{exact no. of days}}{365}.$$

A usual practice, and in some states this is the legal practice, is to count the first day or the last day, but not both. In this text, ordinary interest is used unless exact interest is specified.

When purchasing registered or coupon bonds, the price is given as of the last interest date plus accrued interest, using exact interest.

EXAMPLE 4–23. A \$100, 5%-bond whose last interest payment was December 31 was bought on February 12 next at 98.5 plus accrued (exact) interest. What was the total price?

The total number of days is 31 for January and 12 for the first part of February. Hence the total price is

$$A = \$98.50 + \tfrac{43}{365}\ \$5 = \$98.50 + \$0.59 = \$99.09.$$

EXAMPLE 4–24. Mr. Smith receives \$97.50 from a bank and promises to repay \$100 four months later. What is the corresponding simple interest rate?

The equation of value (4–6) is

$$100 = 97.50\left(1 + \frac{r}{3}\right),$$

from which it is found that $r = 0.0769 = 7.69\%$. Equation (4–7) could be used directly, recognizing $P = \$97.50$:

$$r = \frac{2.50}{(97.5)\frac{1}{3}} = 7.69\%.$$

Simple discount. Simple discount, often referred to as "Banker's discount" or "interest paid in advance," has these characteristics: The

charges are paid in advance by being deducted from the amount that must be repaid, and the amount of this discount D is directly proportional to the product of the amount A that must be repaid by the time t:

$$D = dAt. \tag{4-8}$$

Here d is the constant of proportionality, called the discount rate, and is usually expressed as a decimal fraction per year or $(100d)\%$ per year. The present value P, namely, the amount actually received, is

$$P = A - D = A - Adt = A(1 - dt). \tag{4-9}$$

Equation (4-9) is readily solved for A or d in terms of the other symbols:

$$A = \frac{P}{1 - dt},$$

$$d = \frac{A - P}{At} = \frac{D}{At}. \tag{4-10}$$

Note that the simple discount rate and the simple interest rate (Eq. 4-7) for the same A, P, and t are not equal.

EXAMPLE 4-25. Miss Jones, a school teacher, borrows money from a bank which uses a simple discount rate of 8%. She promises to make payments of \$250 at the end of 3, 4, 5, and 6 months, respectively.* What amount does she receive from the bank?

$$P = \$250[(1 - \tfrac{3}{12} \cdot 8\%) + (1 - \tfrac{4}{12} \cdot 8\%) + (1 - \tfrac{5}{12} \cdot 8\%) + (1 - \tfrac{6}{12} \cdot 8\%)]$$
$$= \$250[4 - \tfrac{18}{12} \cdot 8\%] = \$250(3.88) = \$970.00$$

EXAMPLE 4-26 (compare Example 4-24). Mr. Smith receives \$97.50 from a bank and promises to repay \$100 four months later. What is the corresponding simple discount rate?

The equation of value is

$$97.50 = 100\left(1 - \frac{d}{3}\right),$$

from which it is seen that $d = 7\frac{1}{2}\%$. A comparison with the earlier calculation shows that a simple discount rate of 7.50% for four months and a simple interest rate of 7.69% for four months are equivalent.

---

* A more detailed discussion of partial payment plans is given in Chapter 10 after a discussion of arithmetic and geometric progressions.

More generally, it is possible to find the relation between corresponding simple interest and discount rates using Eqs. (4–6) and (4–10):

$$A = P(1 + rt) = \frac{P}{1 - dt}$$

$$1 + rt = \frac{1}{1 - dt}$$

$$rt = \frac{1}{1 - dt} - 1 = \frac{dt}{1 - dt}$$

so that

$$r = \frac{d}{1 - dt}. \tag{4–11}$$

In the above example, if $t = \frac{1}{3}$ and $d = 7.5\%$:

$$r = \frac{7.5\%}{1 - 0.025} = \frac{7.5\%}{0.975} = 7.69\%.$$

Using either Eqs. (4–7) and (4–9), or directly from Eq. (4–11), we get

$$d = \frac{r}{1 + rt}. \tag{4–12}$$

Problem Set 4–4

1. Find the ordinary and exact simple interest on \$400 from March 1 to each of the following dates, if the rate is 5% per year.

(a) June 1 (b) June 11 (c) June 16 (d) June 19

2. A coupon bond of the given denomination and interest rate, with the last interest paid on June 30, was bought on the date indicated and at the indicated quotation plus accrued (exact) interest. Find the total price.

Denomination	*Interest Rate*	*Quotation*	*Date of Purchase*
(a) \$100	5%	97.6	August 19
(b) \$100	6%	102.5	July 24
(c) \$1000	4%	92.5	September 15
(d) \$1000	8%	104.7	August 26

3. Mr. White borrows money from a bank. He receives \$290 and promises to repay \$300 six months later. What are the corresponding simple discount and simple interest rates?

4. Derive the formula $d = r/(1 + rt)$. Using this formula, check your results in problem 3.

5. Mr. Green borrows $285 net and promises to repay $300 eight months later. What are the corresponding simple discount and simple interest rates? Use Eq. (4–11) as a check.

6. A man borrows $4000 from his insurance company for six months at 6% simple interest. (a) How much is due at the end of that time? (b) He finds he can only repay $2500 on his note when it falls due. He gives a new note for the balance due at 6½% simple interest and repays his debt four months later. How much does he pay at that time?

7. Miss Bowers, a school teacher, borrows money from a bank which uses a simple discount rate of 7½%. She promises to make payments of $150 at the end of 4, 5, 6, and 7 months, respectively. What does she receive from the bank?

8. Miss Jones borrows money from a bank and receives $960. She promises to make payments of $250 at the end of 3, 4, 5, and 6 months, respectively. What is the corresponding simple discount rate?

9. In a certain state, the legal rate on unsecured loans under $100 is 30% simple discount (2½% per month). If Mr. Poor borrows $90 under such a plan: (a) What should he repay at the end of each month for the next three months? (b) What should he repay at the end of each of the third, fourth, and fifth months?

4–6 Linear functions and linear equations in two variables. A *function* is a correspondence or rule which associates with each number x of a given set D *one and only one* number y of a second set R.

The set D is called the *domain* of the function. Unless stated otherwise, we understand that D is the set of all real numbers. The set D may be restricted by the nature of the algebraic formula or by given inequalities. The set R is called the *range* of the function. It is ordinarily determined from the given rule. Examples of functions of the variable x are (a) $3x$; (b) $2x + 1, x \geqq 0$; (c) $x/(x^2 - 1)$; (d) $2x/(x^2 + 1)$. In (a), the domain of x is the set of all real numbers, and the range of the function is also the set of all real numbers. In (b), the domain of x is the set of all positive real numbers and zero, and the range of the function is the set of all positive real numbers greater than or equal to 1. In (c), the domain of x is the set of all real numbers, with the exception of $x = 1$ and $x = -1$. The range of the function is the set of all real numbers. In (d), the domain of x is the set of all real numbers, and the range of the function is the set of real numbers between and including -1 and 1.

The function (or correspondence) may be represented by a single letter such as f, g, h, F, etc., much as letters x, y, a, b are used to represent numbers. To indicate that the rule operates on the variable x in the domain D, the function is written as $f(x)$. The statement $f(x)$, $0 < x < 1$ is read: "the f function of x, x between 0 and 1." A single letter, y, may be used to represent $f(x)$, by setting $y = f(x)$. Here y is a variable restricted to the range R determined by the function f and the domain D of x. If a is a fixed number, $f(a)$ represents the unique number determined when

x is replaced by a in the expression for which $f(x)$ stands. This operation is called *functional evaluation.* Polynomials in x are examples of functions. The simple polynomial

$$mx + b, \quad m \neq 0,$$

with or without restrictions on x, is called a *linear function of* x. For each x there is one and only value of $mx + b$. A variable y is introduced by the defining equation:

$$y = mx + b, \quad m \neq 0, \tag{4–13}$$

and y is called a *linear function of* x. If $y = x$, y is the *identity function of* x, and if $y = b$, y is a *constant function.* The equation

$$Ax + By + C = 0, \quad AB \neq 0, \tag{4–14}$$

where A, B, C are constants while x and y are variables, may be solved for y in terms of x to obtain y *as a linear function of* x. This equation can also be solved for x in terms of y and so it also gives x *as a linear function of* y. Equations (4–13) and (4–14) are called *linear equations in the two variables:* x *and* y. In the graphical representation that follows they represent straight lines.

Graphical representation. Certain arbitrary but convenient conventions are adopted for use throughout the text. Take two mutually perpendicular lines (Fig. 4–3), one horizontal and the other vertical, and let O be their point of intersection. The horizontal line is called the x-axis and the positive direction on it is considered to be to the right; the vertical line is called the y-axis and the positive direction on it is considered to be upward. On each axis consider a number scale (Section 1–7) with O, the origin, assigned the number zero on both lines. When the two variables x and y represent quantities with the same geometric or physical character-

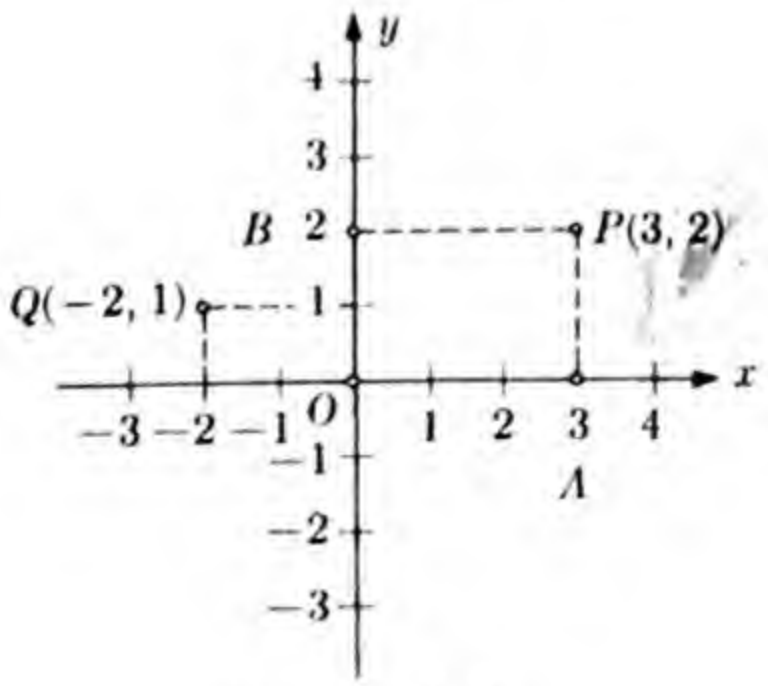


FIGURE 4–3

12348.

istics, or when these are given abstractly as numbers with no indication of their physical interpretation, the same unit of measure is used on both axes. When the variables have different physical characteristics (such as pounds and dollars) the units are selected to fit the particular problem. On each axis there is a 1–1 correspondence between the points and the real numbers.

The points in the plane are set into 1–1 correspondence with the ordered number pairs (x, y) as follows: Through any point P draw a line parallel to OY and intersecting OX at A. Similarly, draw a line through P which is parallel to OX and intersects OY at B. If the coordinate of A is a, and the coordinate of B is b, then P corresponds to the ordered number pair (a, b). Conversely, for the ordered number pair (a, b), points A and B with coordinates a and b are located on the two axes and lines drawn through each parallel to the other axis. These meet at a point P which corresponds to (a, b). The 1–1 correspondence so obtained is called a rectangular coordinate system in the plane and indicated by the notation $P(a, b)$. For the special points A and B, we have $A(a, 0)$ and $B(0, b)$. Figure 4–3 shows the special cases $A(3, 0)$, $B(0, 2)$, $P(3, 2)$, $Q(-2, 1)$.

Equation of a line and the graph of a linear equation. If a variable point P with coordinates (x, y) moves along a curve, an equation which relates the variables x and y is called an equation of the curve. Conversely, for any equation which relates two variables x and y, the totality of points (x, y) whose coordinates satisfy the equation is called the graph of the equation. It is now shown that an equation which corresponds to a straight line is a linear equation and, conversely, the graph of a linear equation is a straight line.

If the line is vertical, let it meet the x-axis at the point $A(a, 0)$. Every point $P(x, y)$ on the line has $x = a$. Conversely, the equation $x = a$ is satisfied for all points (a, y), all of which lie upon the vertical line through the point $A(a, 0)$.

If the line is not vertical, any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on it have $x_1 \neq x_2$. The number m, defined by the equation

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \tag{4–15}$$

is called the slope of the line segment P_1P_2. If the lines through P_1 and P_2 parallel to the axes meet the axes in the points M_1, M_2, N_1, N_2, respectively, and if the lines N_1P_1 and M_2P_2 meet at T, then $x_2 - x_1$ is the directed distance $M_1M_2 = P_1T$, and $y_2 - y_1$ is the directed distance $N_1N_2 = TP_2$. The slope of segment P_1P_2 is the ratio of the change in y to the change in x, where directed distances are taken into account. If $P(x, y)$ is any other point on the nonvertical line through P_1 and P_2,

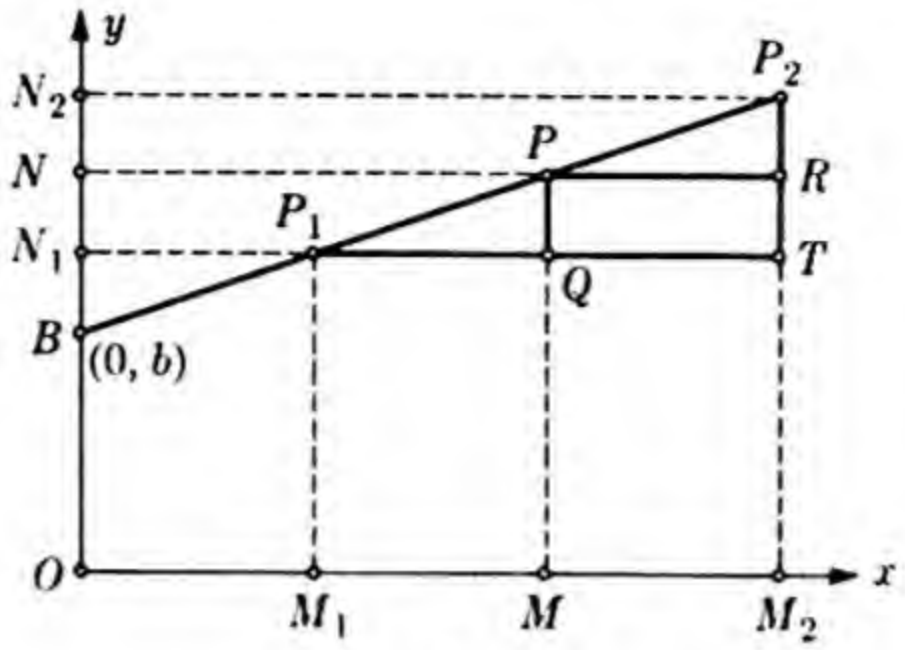


FIGURE 4–4

use lines through P parallel to the axes to determine the points M, N, Q, and R as in Fig. 4–4. Then the slope of the segment P_1P is

$$m' = \frac{y - y_1}{x - x_1} = \frac{N_1N}{M_1M} = \frac{QP}{P_1Q},$$

and

$$m = \frac{TP_2}{P_1T}.$$

Properties of parallel lines and similar triangles show that $m' = m$ for all points P. Hence m, called the slope of the line, is independent of the points used. Conversely, if P_1, P_2 and P are three given points and the slopes of the segments P_1P_2 and P_1P are equal, the three points are collinear. The line is completely determined by the point P_1 and the slope m. If P is any point of the line different from P_1, then

$$\frac{y - y_1}{x - x_1} = m, \qquad (x \neq x_1)$$

and

$$y - y_1 = m(x - x_1). \tag{4–16}$$

This equation is also satisfied by the point $P_1(x_1, y_1)$ and hence is the equation of the nonvertical line through $P_1(x_1, y_1)$ with slope m. It is referred to as the *point-slope form* of the equation of the line.

Let the nonvertical line meet the y-axis in the point $B(0, b)$. The directed distance OB, the number b, is called the *y-intercept*. If this special point is used instead of the general point P_1, Eq. (4–16) becomes

$$y = mx + b. \tag{4–17}$$

This is the *slope-y-intercept form* of the equation of the line.

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points whose coordinates satisfy the equation $y = mx + b$, then

$$y_2 = mx_2 + b$$

$$\underline{y_1 = mx_1 + b}$$

$$y_2 - y_1 = m(x_2 - x_1)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = m, \qquad (x_2 \neq x_1).$$

Since any two points that satisfy the equation determine the same slope, the graph of Eq. (4–17) is a line.

The equation of a vertical line and that of a nonvertical line are both covered by the linear equation

$$Ax + By + C = 0, \quad (A \text{ and } B \text{ not both zero}), \qquad (4\text{–}18)$$

and this equation is called the *general form* of the equation of a line. Conversely, the linear equation

$$Ax + By + C = 0, \quad (A \text{ and } B \text{ not both zero}),$$

reduces to the form $x = a$ if $B = 0$. If $B \neq 0$, it reduces to the form $y = mx + b$. Hence the graph of the linear equation, Eq. (4–18), is a line. Therefore, the equation of a straight line is a linear equation and the graph of a linear equation is a straight line.

The graph of the equation of a line may be found from any two points whose coordinates satisfy the equation. Any two values of x can be used to compute the corresponding values of y. The points are then plotted and the line drawn. A third point is often used as a check. Important points to find are the intercepts on the axes. These are obtained by putting $x = 0$ and solving for y, and then putting $y = 0$ and solving for x. If the line goes through the origin, these give the same point and another point is needed. If the equation is given in the form (4–16) or (4–17), one point $(0, b)$ is known directly and a second point can be found from the slope. Write m as p/q. To obtain a second point, start at the known point, move horizontally a directed distance q and then vertically a directed distance p.

EXAMPLE 4–27. Write the equation $5x + 3y = 18$ in different forms and find its graph.

Solve the equation for y:

$$y = -\tfrac{5}{3}x + 6.$$

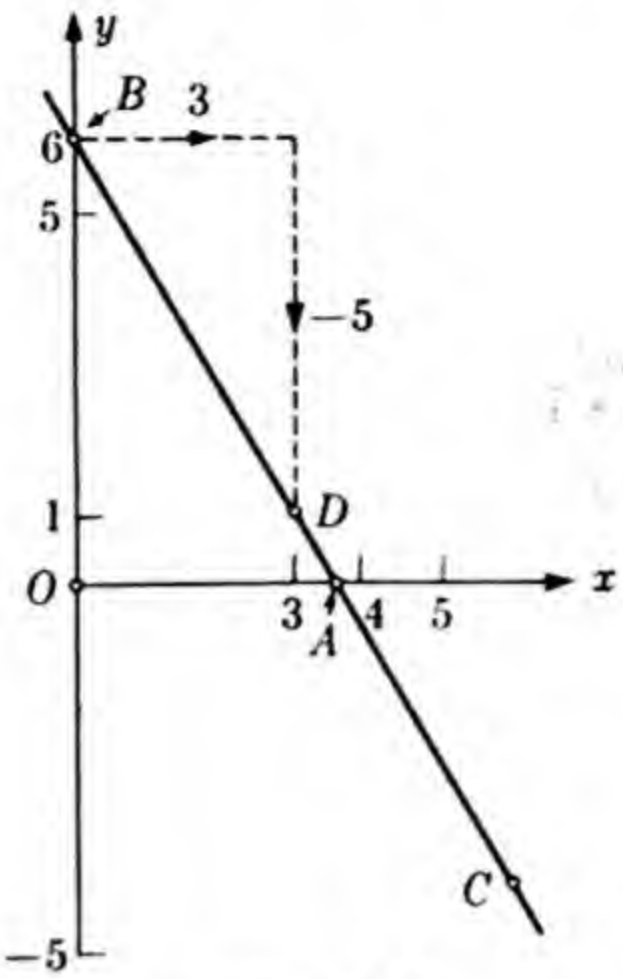


FIGURE 4–5

In this equation, the value $x = 6$ gives $y = -4$. The Y-intercept is 6 and the slope of the line is $-5/3$. If the original equation is divided by 18, we find

$$\frac{5}{18}x + \frac{3}{18}y = 1$$

or

$$\frac{x}{18/5} + \frac{y}{6} = 1.$$

In this, if $x = 0$, $y = 6$; if $y = 0$, $x = 18/5$, and these are the two intercepts. Three points on the line: $B(0, 6)$, $A(18/5, 0)$, and $C(6, -4)$, are available to draw the line and for checking (Fig. 4–5).

Since the slope of the line is $-5/3$, the point D is obtained by starting at B, moving 3 units to the right and 5 units downward.

PROBLEM SET 4–5

1. If a function $f(x)$ is defined as below, find $f(2)$, $f(y)$, $f(2 + y)$, $f(x + k)$.

(a) $f(x) = 3x - 4$

(b) $f(x) = \dfrac{3x - 4}{x + 2}$, $(x \neq -2)$

(c) $f(x) = \dfrac{2x}{x^2 + 1}$

(d) $f(x) = x^2 - 2x + 4$

2. For each of the following linear functions, determine the domain of x and the range of the function of x.

(a) $f(x) = 3x + 2$

(b) $f(x) = 3x + 2$, $(x \geqq 0)$

(c) $f(x) = 3x + 2$, $[f(x) \geqq 0]$

3. For each of the following linear functions, determine the domain of x and the range of $f(x)$.

(a) $f(x) = 3x - 2, (x \geqq 0)$ (b) $y = f(x) = 3x - 2, (y \geqq 0)$

(c) $y = f(x) = 3x - 2, (x \geqq 0, y \geqq 0)$

4. Graph each of the following lines from the given point and given slope. Find the equation of each line and determine its y-intercept. Check the graph.

(a) $P(2, 3), m = 1/3$ (b) $P(2, 3), m = -3$

(c) $P(-3/2, 5/2), m = -1$ (d) $P(-2, -4), m = 2$

5. Graph each of the following lines from the given y-intercept and slope. Find the equation of each line and its x-intercept. Check the graph.

(a) $b = 4, m = 2$ (b) $b = -5/2, m = 5/2$

(c) $b = 10, m = -3/2$ (d) $b = -2, m = -2/3$

6. Find the graphs of the following equations; use a third point as a check point.

(a) $y = \frac{3}{2}x + 3$ (b) $y = \frac{2}{5}x$

(c) $y = -\frac{2}{3}x + 3$ (d) $y = \frac{5}{3}x - 6$

(e) $y = -2x + 5, (0 \leqq x \leqq 2)$ and $y = 1, (2 \leqq x \leqq 4)$

(f) $y = 2x - 3, (0 \leqq x \leqq 3)$ and $x = 3, (0 \leqq y \leqq 6)$

7. Use the intercepts to find the graphs of the following equations. Use other points if needed.

(a) $3x + 4y = 24$ (b) $3x - 4y = 0$

(c) $3x - 4y = 12$ (d) $5x + 8y = 12$

(e) $x + y + 1 = 0$ (f) $8x - 5y + 160 = 0$

4–7 Applications. In applications of linear equations to physical or financial problems, the variables often have entirely different natures, so the same unit cannot be used on both axes. The units are selected so that the significant part of the line appears in the diagram. If the variables are x and y, the changes in y and x are measured with different units. The ratio of change in y to change in x is referred to as the *relative slope* in order to distinguish it from the slope of a line when one standard unit is used to measure all distances. When no confusion results, the word "relative" is dropped. For example, in the distance formula for uniform motion, $d = rt$, t may be expressed in hours, d in miles, and the rate of change of d with respect to t is then expressed in miles per hour. Graphically, r represents the relative slope of the line, where the horizontal axis is used for the time scale, the vertical axis is used for the distance scale, and only lengths parallel to either axis have physical significance.

EXAMPLE 4–28. In traveling between two towns 150 mi apart, a freight train moves at the rate of 30 mph for the first 50 mi, is then sidetracked

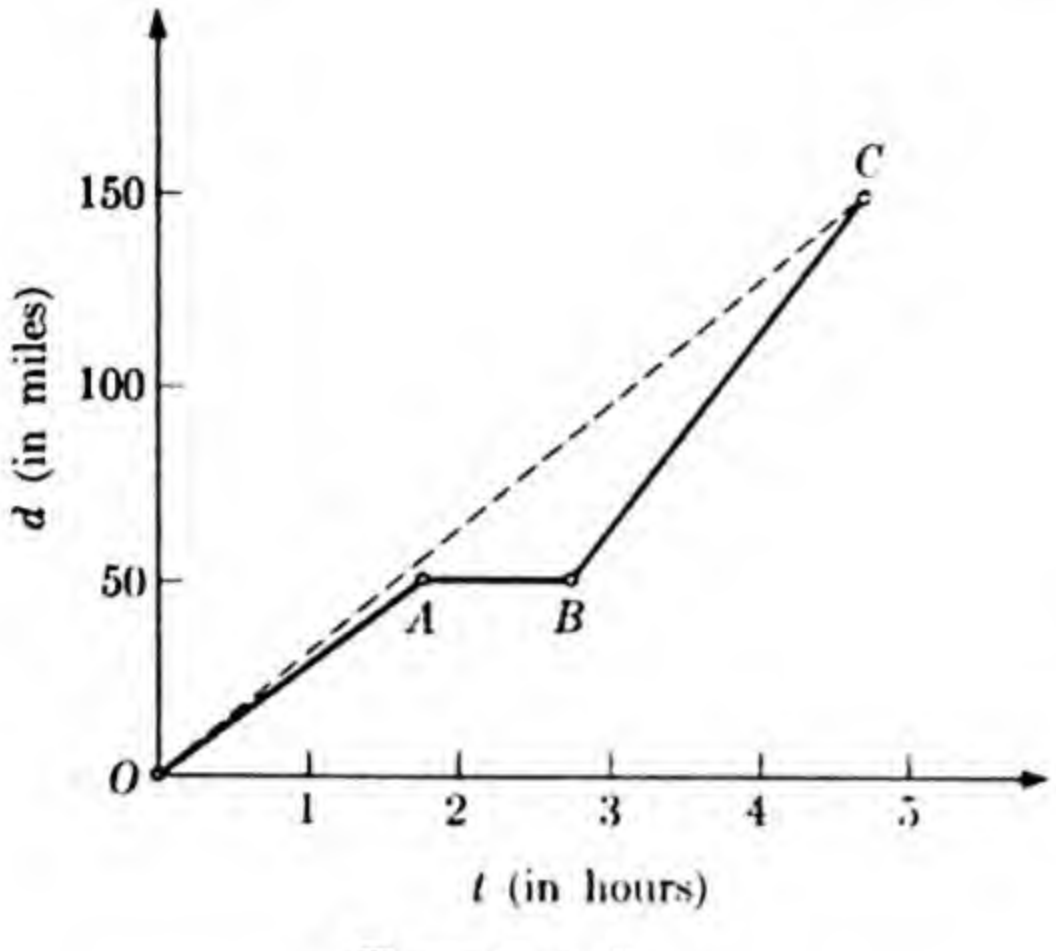


FIGURE 4–6

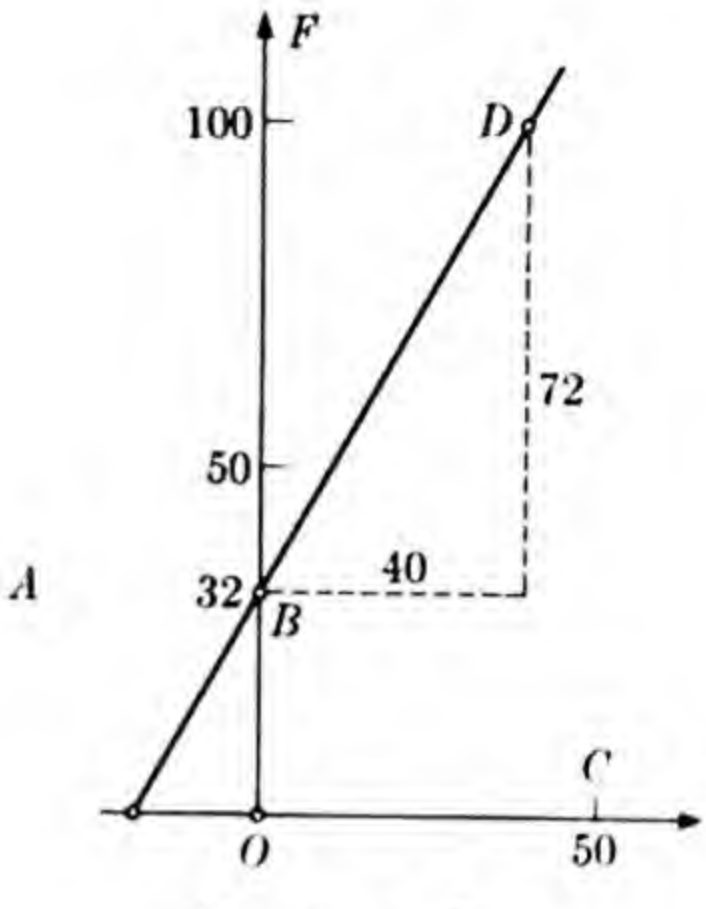


FIGURE 4–7

for an hour, and then continues its journey at the rate of 50 mph. What is its average speed?

The distance scale is adapted to the 150 mi involved and a rough calculation indicates that the time range is of the order of magnitude of 5 hr. The time unit is chosen so that the diagram fits in the desired space. The equation for the first part of the journey is $d = 30t$, $(0 \leqq d \leqq 50)$. The corresponding line segment OA (Fig. 4–6) is found by starting from O and drawing a line with relative slope of 30 (rise 30, run 1) until it reaches a vertical distance of 50 at the point A. The line segment AB has the equation $d = 50$, $(\frac{5}{3} \leqq t \leqq \frac{8}{3})$, and extends to $B(\frac{8}{3}, 50)$. From B, the graph continues along a segment of relative slope 50 until it reaches a point C for which $d = 150$ and $t = \frac{14}{3}$. The equation of the line segment BC is found from Eq. (4–16):

$$(d - 50) = 50\left(t - \frac{8}{3}\right),$$

$$\left(\frac{8}{3} \leqq t \leqq \frac{14}{3}\right).$$

The average speed is the relative slope of the line OC, namely,

$$150 \div \frac{14}{3} = \frac{450}{14} = 32\tfrac{1}{7} \text{ mph}.$$

EXAMPLE 4-29. Represent graphically the relation between the Centigrade and Fahrenheit temperature scales:

$$F = \frac{9}{5}C + 32, \qquad (0 \leqq F \leqq 100).$$

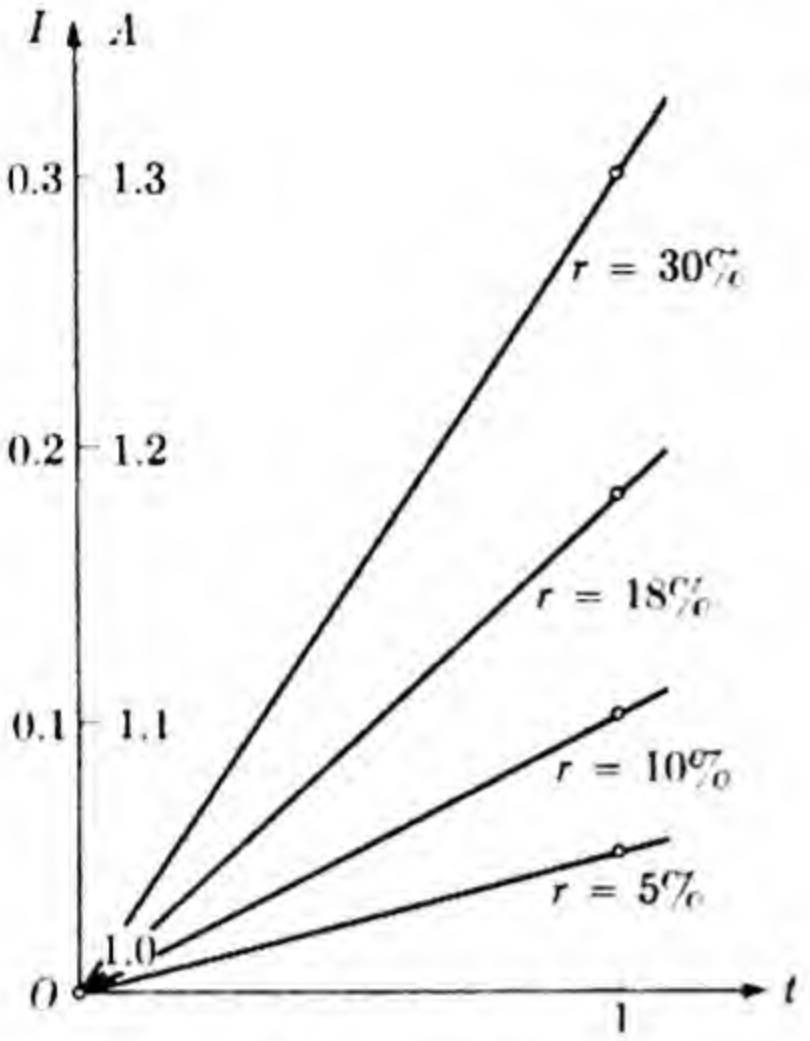


FIGURE 4–8

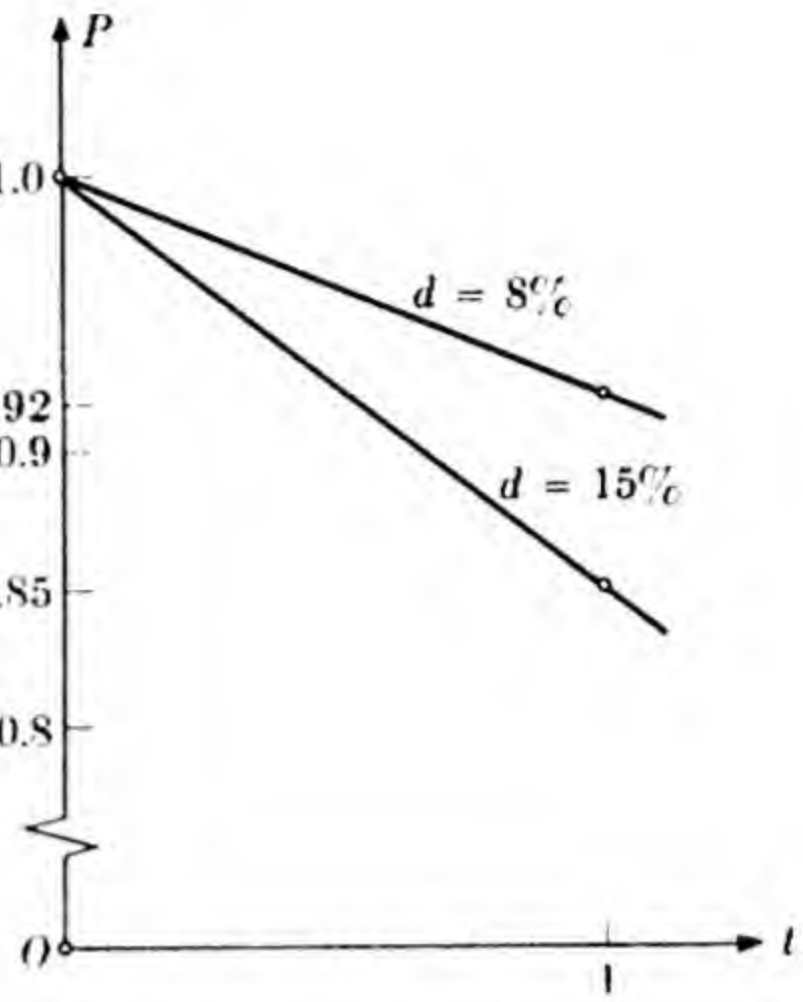


FIGURE 4–9

The units may be chosen of equal sizes but such that $F = 100$ appears in the diagram. The line is sketched from its intercepts:

$$C = 0, \quad F = 32; \quad F = 0, \quad C = -\tfrac{160}{9}$$
$$= -18 \text{ (approx.)}$$

The slope, $\frac{9}{5} = \frac{72}{40}$, provides a check Fig. (4–7).

EXAMPLE 4–30. The laws of simple interest for (the principal) $P = 1$, take the form

$$I = rt \qquad \text{and} \qquad A = 1 + rt.$$

If r is a constant, the graph of each of these equations is a straight line, the first passing through the origin and the second with y-intercept 1. The rate r is usually a small decimal fraction and t is usually less than 1, so the scales may be given in small units. To show A and I on the same diagram, the origin of the A-scale is not shown, but the A-scale is the I-scale with each reading increased by 1, and the same line can be used for either equation. Figure 4–8 shows such lines for $r = 5\%$, 10%, 18%, and 30%.

Similarly, if the amount subject to simple discount is $A = 1$, the present value is given by the equation

$$P = 1 - dt.$$

In order that the corresponding line should appear in the diagram, only the upper part of the P-scale is shown. The omission is indicated by the use of a zig-zag line. Figure 4–9 shows the lines for $d = 8\%$ and 15%.

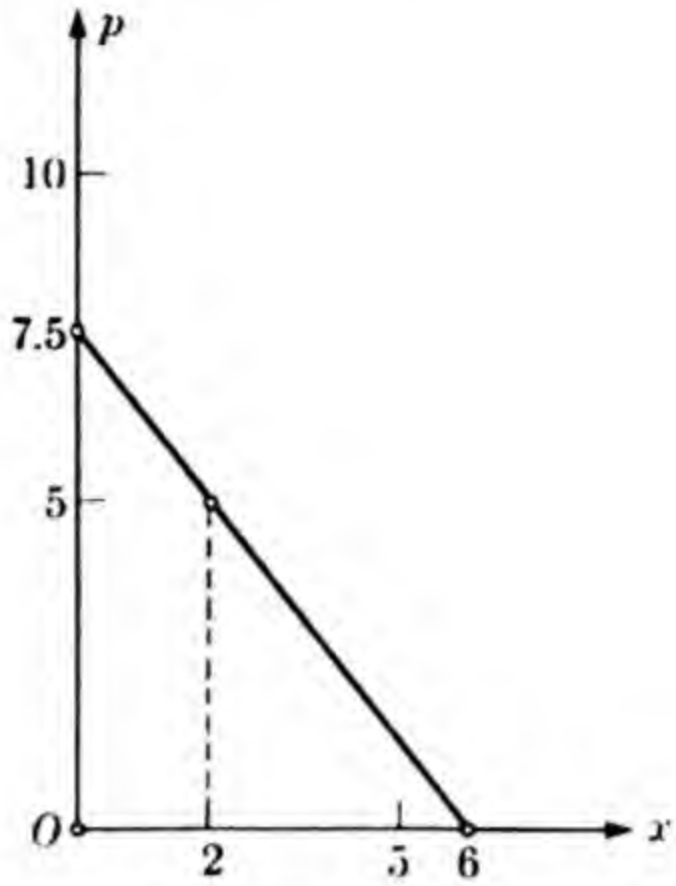


FIGURE 4–10

Linear laws of demand and supply. In a competitive market, there are commodities whose average price p for each unit of the commodity depends upon the number of units x of the commodity demanded by the consumers. "All other things being equal," the price p per unit of the commodity demanded may be considered a monotonically decreasing function of x, that is, as x increases, p decreases. The simplest such function is a linear function, and the graph of the equation is part of a straight line for which both p and x are zero or positive. Such equations take one of the forms

$$\begin{gathered} p = p_0 + mx, \\ x = x_0 - kp, \\ Ax + Bp = C, \quad (x \geqq 0, p \geqq 0), \end{gathered} \tag{4–19}$$

where p_0 is the p-intercept and is the highest price that is paid for the commodity; x_0 is the x-intercept and is the quantity demanded when the commodity is free; m, a negative number, is the slope of the line, and k is the negative reciprocal of m. Any one of these forms can be reduced to the other forms by simple algebraic processes.

EXAMPLE 4–31. A demand equation is given in the form $5x + 4p = 30$, $(x \geqq 0, p \geqq 0)$. Use the intercepts to draw the corresponding line segment. These intercepts correspond to the highest price that will be paid for the commodity $(x = 0)$ and the largest amount that will be demanded $(p = 0)$. Find a check point from the slope of the line.

Solve for p:

$$p = -\tfrac{5}{4}x + \tfrac{15}{2}.$$

The highest price any one would pay under the assumed law is $p = \frac{15}{2}$. The maximum quantity demanded is found from the equation to be $x = 6$.

From these two intercepts, the line segment can be drawn. The slope of the line is $-(5/4)$, which is interpreted as $5 \div (-4)$. Starting at the x-intercept, decrease x by 4 and increase p by 5 to get a check point (2, 5) on the line (Fig. 4–10).

In a competitive economy, the supplier of certain commodities increases the average price p for each unit of the commodity supplied when the quantity x to be supplied increases. "All other things being equal," the price p per unit of the commodity supplied may be considered a monotonically increasing function of x, that is, as x increases, p increases. The simplest such function is a linear function, and the graph of the equation is part of a straight line for which p and x are zero or positive. Such equations take one of the forms

$$\begin{aligned} p &= P_0 + Mx, \\ x &= Kp - X_0, \\ Ax + Bp &= C, \quad (0 \leqq x \leqq L, p \geqq 0), \end{aligned} \tag{4–20}$$

where P_0 is the lowest price at which the supplier will offer the commodity; M is the slope of the line and K is its reciprocal; X_0 is a positive number, indicating that the line has a negative x-intercept. The part of the line corresponding to negative values of x has no economic significance. Any one of these forms reduces to the other forms by simple algebraic processes.

EXAMPLE 4–32. A supply equation is given in the form

$$15x - 4p + 50 = 0, (0 \leqq x \leqq 10).$$

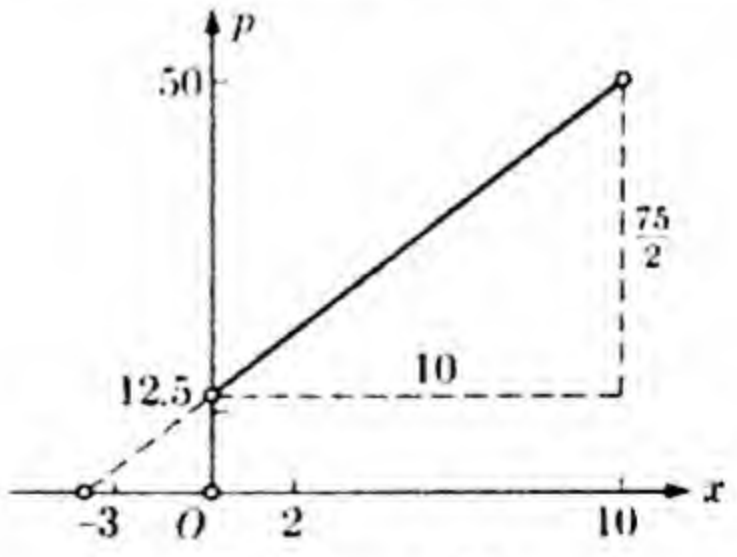
Use the intercepts and the point for which $x = 10$ to determine appropriate scales. Draw the corresponding line segment, including a check point. Point out the part of the line that is economically significant.

Solve the given equation for p:

$$p = \tfrac{15}{4}x + \tfrac{50}{4}.$$

The three points used are

x	0	10	$-10/3$
p	12.5	50	0



FIGUGE 4–11

These data indicate the size of the x-unit and the p-unit. The part of the line for which x is negative is not economically significant, but includes the x-intercept which is used as a check. The slope of the line is $75/2 \div 10 = 15/4$ (Fig. 4–11).

Budget equation. If a given expendable income E is to be spent on two commodities, and the amounts of the commodities are x and y while their respective prices are A and B, then the equation

$$Ax + By = E, \tag{4–21}$$

with all quantities positive, is called the budget equation for two commodities. Its graph is that part of a straight which lies in the first quadrant.

EXAMPLE 4–33. Draw the budget curve for prices of 3 and 4, with an expendable income of 25.

The equation is

$$3x + 4y = 25,$$

$$(x \geqq 0, y \geqq 0).$$

The line is determined (Fig. 4–12) from two of the three points

x	0	$8\frac{1}{3}$	3
y	$6\frac{1}{4}$	0	4

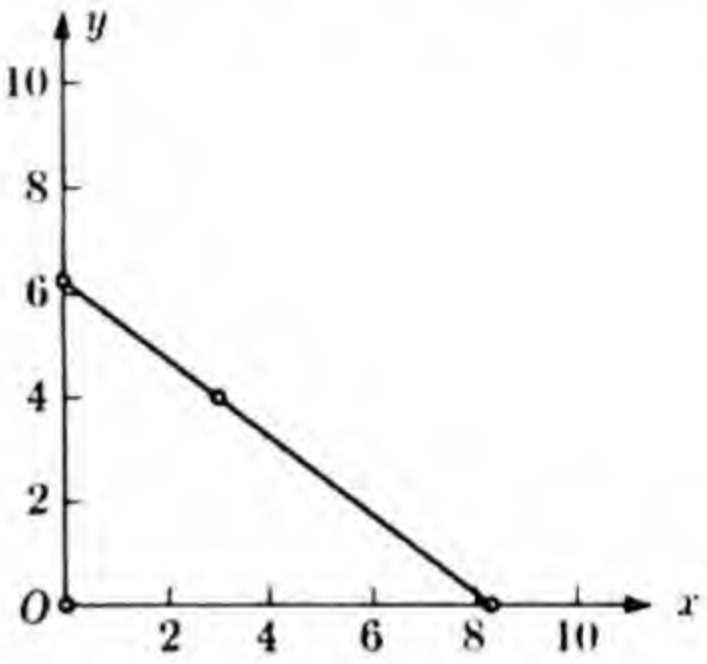


FIGURE 4–12

PROBLEM SET 4–6

1. In traveling between two stations, 100 mi apart, a freight train moves at the rate of 25 mph for the first 40 mi; it is delayed for half an hour and then continues its journey at the rate of 40 mph. Represent these data by means of a broken line (see Fig. 4–6) and write the equations of the three different segments. What is the value and the geometric significance of its average speed for the total elapsed time?

2. A student drives 45 mi to school and allows $1\frac{1}{2}$ hours for the trip. One day he made the first 30 mi at the average rate of 40 mph. How fast should he drive the remaining distance to arrive just on time? Represent these data and results graphically, writing equations for the corresponding line segments.

3. Represent graphically the relation between the Fahrenheit and Centigrade temperature scales

$$C = \tfrac{5}{9}(F - 32), \qquad (0 \leqq C \leqq 100).$$

4. Mr. Poor borrows money from a loan company which uses a simple discount rate of $2\frac{1}{2}\%$ per month. If he borrows the money for T months, how much does he receive for each dollar he borrows? Draw the corresponding graph for $0 \leqq T \leqq 12$.

5. If the demand law is $p = 45 - 5x$, (a) what is the highest price anyone would pay for the commodity, and (b) what is the quantity demanded if the commodity is free? Draw the demand curve. (c) If the given demand law

were restricted to the domain $0 \leqq x \leqq 7$ and if p remained constant thereafter until $x = 10$, what is the nature of the demand curve?

6. The demand law is $x = 64 - 4p$. (a) What is the highest price anyone would pay for the commodity? What is the amount demanded if the commodity is free? Draw the demand curve. (b) If the given demand is valid until the price drops to $p = 4$, and remains fixed until $x = 80$, what is the nature of the demand curve? Write the corresponding equations which give p as a function of x, indicating appropriate limitations on x.

7. A demand law is given in the form $4x + 9p = 48$. (a) Find the intercepts from this form and draw the corresponding curve. (b) Solve this equation for p in terms of x and find the slope of the line. Use this slope to locate a third point on the line.

8. The supply function is $p = 2\frac{1}{2} + \frac{3}{4}x$, $(0 \leqq x \leqq 12)$. Draw the supply curve, using the values $x = 0$ and $x = 12$. Find the x-intercept and check the graph.

9. The supply function is $x = \frac{4}{3}p - 4$, $(0 \leqq p \leqq 9)$. Draw the graph from its intercepts and use the value $p = 9$ to determine a third point. What part of the curve is significant?

10. The supply function is given by the equations $2p - 3x = 6$, $(0 \leqq x \leqq 4)$ and $3p - 2x = 19$, $(4 \leqq x \leqq 9)$. Draw the supply curve.

11. The total cost Q of producing x units of a certain commodity is

$$Q = 4 + \tfrac{1}{8}x, \qquad (0 \leqq x \leqq 40).$$

For $40 \leqq x \leqq 100$ the cost of each additional unit is $\frac{1}{10}$. Find the equation for Q in this new interval. Find the equation of the line segment joining the points corresponding to $x = 0$ and to $x = 100$. Show all three line segments in the same diagram.

12. Draw the budget curve for prices of 4 and 7 with an expendable income of 35. What is the slope of the line? Use this slope to find a check point.

13. If a given expendable income E is to be spent on three commodities, and the amounts of the commodities are x, y, and z with their corresponding prices of A, B, and C, what is the budget equation? How could you represent this graphically? If there were four commodities, what could be done?

CHAPTER 5

SIMULTANEOUS LINEAR EQUATIONS

5–1 Simultaneous linear equations in two variables. Portions of this section apply to any two equations in two variables, although the treatment is given for two linear, conditional equations in two variables. These are written in the form:

$$A_1x + B_1y = C_1, \tag{5–1}$$

$$A_2x + B_2y = C_2. \tag{5–2}$$

An ordered number pair (x_0, y_0) which satisfies both of these equations is called a solution of the simultaneous Eqs. (5–1) and (5–2). The totality of such ordered number pairs is called the solution set. When there is no number pair which satisfies the equations, the solution set is said to be the null set. Unless otherwise stated, the domain of the variables is understood to be the set of all real numbers. In problems having economic significance, the variables may be restricted to the non-negative real numbers.

Two pairs of linear equations are said to be *equivalent* when they have the same solution sets. Arithmetical operations are performed on a pair of linear equations to obtain other pairs of equations which are equivalent to the original pair. The process of solving a given pair of simultaneous linear equations consists of finding equivalent pairs of equations in sequence until a pair is obtained for which the solution set is recognized. An operation on the given equations which leads to an equivalent set of equations is called *permissible*. The permissible operations of Section 4–1 on either of the two equations may be used. These are: (1) removal or insertion of parentheses, (2) addition of the same algebraic expression to both members of an equation (transposition being a special case), and (3) multiplication of both members of an equation by the same nonzero constant. Additional permissible operations which combine the two equations are: (4) substitution and (5) elimination by addition and subtraction. These are used in the algebraic solutions of Eqs. (5–1) and (5–2).

5–2 Algebraic solution. *Method of substitution.* If $B_1 = 0$, solve Eq. (5–1) for x and substitute this value, $x = x_0$, in Eq. (5–2). The resulting equation,

$$A_2x_0 + B_2y = C_2, \tag{5–3}$$

and the equation $x = x_0$ form a pair that is equivalent to the original pair.

If Eq. (5–3) has the solution $y = y_0$, then the pair $x = x_0$, $y = y_0$ is also equivalent to the given system and gives the solution set.

If $B_1 \neq 0$, solve Eq. (5–1) for y and substitute this value of y,

$$y = mx + b, \tag{5–4}$$

in Eq. (5–2):

$$A_2x + B_2(mx + b) = C_2. \tag{5–5}$$

If Eq. (5–5) has the solution $x = x_0$, then the equations $x = x_0$ and $y = mx + b$ are equivalent to Eqs. (5–1) and (5–2), and y_0 is obtained by substitution to give the solution in the form

$$x = x_0, \qquad y_0 = mx_0 + b.$$

Elimination by addition. If two linear equations are added to get a third linear equation, then this together with either of the original equations gives a pair equivalent to the original pair. If (x_0, y_0) is a solution of Eqs. (5–1) and (5–2), then

$$A_1x_0 + B_1y_0 = C_1,$$

$$A_2x_0 + B_2y_0 = C_2,$$

and by addition:

$$(A_1 + A_2)x_0 + (B_1 + B_2)y_0 = C_1 + C_2.$$

Hence (x_0, y_0) is also a solution of

$$(A_1 + A_2)x + (B_1 + B_2)y = C_1 + C_2. \tag{5–6}$$

By subtraction, if (x_0, y_0) is a solution of Eqs. (5–1) and (5–6), it is also a solution of Eq. (5–2). Hence the pair (5–1) and (5–2) is equivalent to the pair (5–1) and (5–6).

Multiply Eq. (5–1) by B_2 and Eq. (5–2) by $-B_1$ and add to eliminate y. Multiply Eq. (5–1) by $-A_2$, and Eq. (5–2) by A_1 and add to eliminate x. This pair of new equations is equivalent to the original pair. The process can be abbreviated as indicated below:

$$\begin{array}{rl} A_1x + B_1y = C_1 & \left| \begin{array}{r} B_2 \\ -B_1 \end{array} \right| \begin{array}{r} -A_2 \\ A_1 \end{array} \\ \underline{A_2x + B_2y = C_2} & \\ (A_1B_2 - B_1A_2)x = C_1B_2 - B_1C_2 & \end{array} \tag{5–7}$$

$$(A_1B_2 - B_1A_2)y = A_1C_2 - C_1B_2. \tag{5–8}$$

If $(A_1B_2 - B_1A_2) \neq 0$, solve these equations for x and y and, since the

set Eqs. (5–7) and (5–8) are equivalent to the original set, Eqs. (5–1) and (5–2), the unique solution is found. If $(A_1B_2 - B_1A_2) = 0$, special consideration is necessary. This is discussed in the next section.

EXAMPLE 5–1. Solve the simultaneous equations $2x + 4y = 11$, $-5x + 3y = 5$ by the method of substitution and by the method of elimination by addition.

Solve the first equation for x and substitute in the second:

$$x = \frac{11 - 4y}{2}$$

$$-5\left(\frac{11 - 4y}{2}\right) + 3y = 5$$

$$-55 + 20y + 6y = 10$$

$$26y = 65$$

$$y = \frac{65}{26} = \frac{5}{2}.$$

Substitute this value for y into the first equation:

$$2x + 10 = 11$$

$$x = \tfrac{1}{2}.$$

The solution obtained by first eliminating y and then x can be put in the form

$$\begin{array}{rrr|l} 2x + 4y = 11 & & 3 & 5 \\ -5x + 3y = 5 & & -4 & 2 \\ \hline 26x = 13, & & x = \tfrac{1}{2}, & \\ 26y = 65, & & y = \tfrac{5}{2}, & \text{as before.} \end{array}$$

Second-order determinants. The square array of numbers written between vertical bars,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

to which is assigned the value $ad - bc$, is called a *second-order determinant.* Thus

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If the determinant

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = A_1B_2 - A_2B_1$$

is not zero, Eqs. (5–7) and (5–8) take the forms

$$x = \frac{\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}. \tag{5–9}$$

Provided the equations have first been written in the precise form of Eqs. (5–1) and (5–2), the denominator in each case is the determinant formed from the coefficients of x and y. The numerator for x is the determinant formed from the denominator by replacing the A column by the C column, and the numerator for y is the determinant formed from the denominator by replacing the B column by the C column.

5–3 Graphical solution. Dependent and inconsistent equations. The simultaneous linear equations may be solved graphically by drawing the corresponding lines in the same diagram and determining the coordinates of their point of intersection. In general, there will be one and only one solution. In special cases the lines coincide, in which event they have an infinite number of points in common. If the lines are parallel, they have no point of intersection. If two linear equations differ only in that one is obtained from the other by multiplying by a nonzero constant, they are equivalent and have the same line as their graph. In that case the equations are *dependent*. From the equations

$$A_2 = kA_1, \qquad B_2 = kB_1, \qquad C_2 = kC_1, \tag{5–10}$$

it is seen that all the coefficients in Eqs. (5–7) and (5–8), as well as the determinants in Eqs. (5–9), are equal to zero. Equations (5–9) would be meaningless.

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ kA_1 & kB_1 \end{vmatrix} = k(A_1B_1 - A_1B_1) = 0$$

and similarly

$$\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = 0.$$

If in Eqs. (5–1) and (5–2), B_1 and B_2 are both zero, the lines are vertical and parallel (or identical). If $B_1B_2 \neq 0$, and if

$$A_2 = kA_1, \qquad B_2 = kB_1, \qquad C_2 \neq kC_1, \tag{5–11}$$

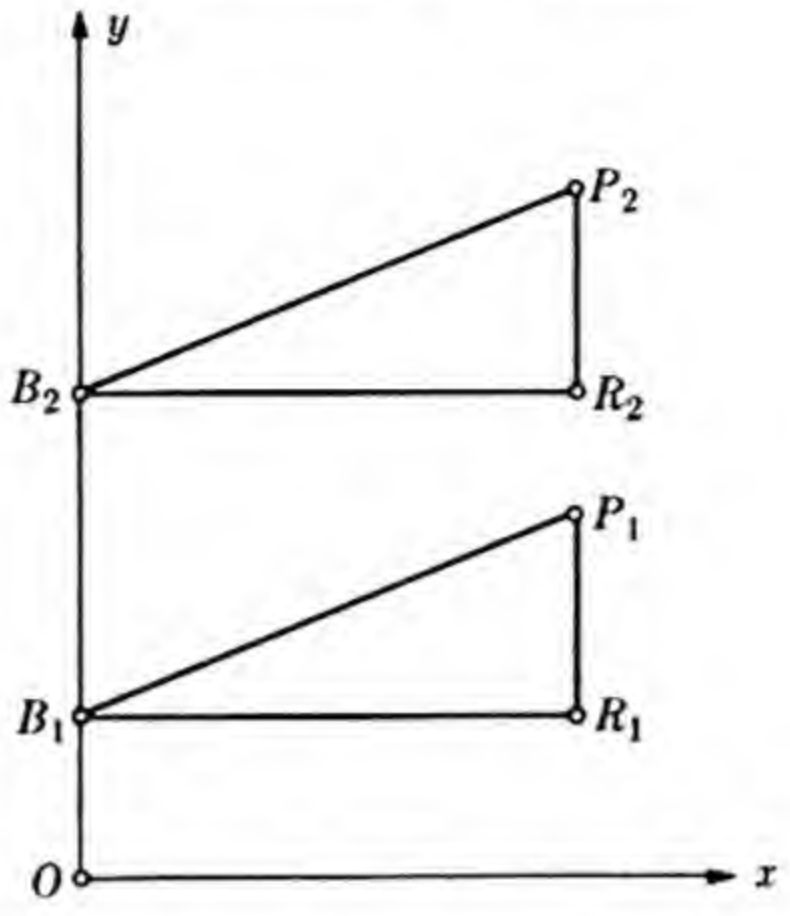


FIGURE 5–1

the two given equations take the forms

$$y = -\frac{A_1}{B_1}x + \frac{C_1}{B_1}; \qquad y = -\frac{A_2}{B_2}x + \frac{C_2}{B_2},$$

with slopes $m_1 = -A_1/B_1$, $m_2 = -A_2/B_2$. It is noted that two non-vertical lines that are parallel have the same slopes and, conversely, if two lines have the same slopes they are parallel. Equations (5–11) show that $m_1 = m_2$ and that the y-intercepts are different. Hence the lines are not coincident but parallel and have no point of intersection (Fig. 5–1). But Eqs. (5–11) also show, as above, that

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} \neq 0.$$

Two linear equations which satisfy Eqs. (5–11) are *inconsistent* and have no solution.

EXAMPLE 5–2. Solve the equations $2x + 4y = 11$, $-5x + 3y = 5$ graphically and by means of determinants.

Both lines are drawn from their intercepts (Fig. 5–2).

$2x + 4y = 11$

x	y
0	11/4
11/2	0

$-5x + 3y = 5$

x	y
0	5/3
−1	0

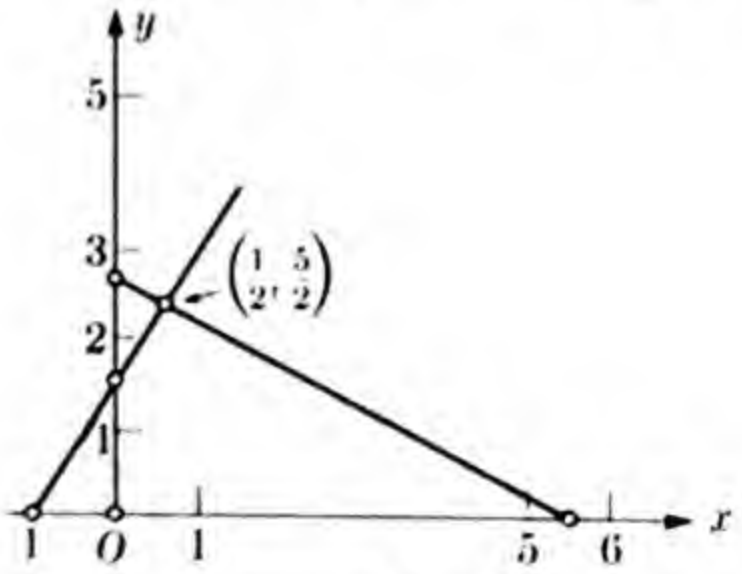


FIGURE 5–2

The solution by determinants is

$$x = \frac{\begin{vmatrix} 11 & 4 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -5 & 3 \end{vmatrix}} = \frac{13}{26} = \frac{1}{2}; y = \frac{\begin{vmatrix} 2 & 11 \\ -5 & 5 \end{vmatrix}}{26} = \frac{65}{26} = \frac{5}{2}.$$

This agrees with the geometrical solution.

EXAMPLE 5–3. Determine the constant A so that the lines $3x - 4y = 12$ and $Ax + 6y = -9$ are parallel.

Multiply the first equation by $-3/2$ in order to make the coefficient of y equal to 6. The new coefficient of x is $-9/2$, so $A = -9/2$ and the required equation is $-\frac{9}{2}x + 6y = -9$ or $3x - 4y = 6$.

PROBLEM SET 5–1

1. Solve the following pairs of simultaneous equations graphically and algebraically, using the method of substitution.

(a) $y = 3x - 4$
$2x - y = 1$

(b) $y = 3x + 4$
$6x - y = 2$

(c) $x + 4y = -3$
$3x - 5y = 25$

(d) $3x - 2y = 1$
$2x + 3y = -8$

(e) $3x + 4y = 12$
$5x + 12y = 60$

(f) $4x + 3y = 10$
$12x - 5y = 9$

(g) $y = \frac{2}{3}x + 4$
$2x - 3y + 12 = 0$

(h) $y = \frac{2}{3}x + 4$
$2x - 3y + 6 = 0$

2. Solve problems 1(a)–1(h) by the method of elimination.

3. Solve the following pairs of simultaneous equations by determinants and check graphically. Do not use the point corresponding to your answer to graph either line, since this would destroy your check.

(a) $2x - 3y = 5$
$5x - 4y = 2$

(b) $2x - 3y = 2$
$3x - 2y = 6$

(c) $2x + 3y = 8$
$6x + 5y = 24$

(d) $3x - 2y = 6$
$6x + 5y = 3$

(e) $4x - 2y = 7$
$6x - 8y = 5$

(f) $5x + 4y = 1$
$8x + 6y = 1$

4. The following pairs of simultaneous equations are either dependent or inconsistent; determine which. Attempt to solve each pair by the method of elimination or by determinants and explain the results. Check the results graphically.

(a) $y = 3x - 4$
$y = 3x + 6$

(b) $y = \frac{3}{4}x + 6$
$3x - 4y + 24 = 0$

(c) $2x - 3y = 8$
$6x - 9y = 18$

(d) $3x - 2y = 8$
$6x - 4y = 16$

(e) $y = \frac{4}{3}x - 8$
$4x - 3y = 16$

(f) $\frac{2}{3}x - \frac{3}{4}y = 2$
$\frac{1}{9}x - \frac{1}{8}y = \frac{4}{3}$

5–4 Equation of a line determined by two conditions. In Eq. (4–17), the equation of a line is given in the form $y = mx + b$, where x and y are the variables, m is the slope of the line, and b is its y-intercept. If m and b are known, the equation of the line is determined. However, m and b may be treated as unknowns that must be found from given conditions. If m and a point (x_1, y_1) on the line are given, b is found from the equation

$$y_1 = mx_1 + b.$$

The equation of the line under these conditions is

$$y = mx + y_1 - mx_1, \qquad y - y_1 = m(x - x_1).$$

When two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ are given, the equation of the line is determined from the two linear equations

$$y_1 = mx_1 + b, \qquad y_2 = mx_2 + b$$

in m and b. These may be solved by any of the given methods. The method of determinants yields

$$m = \frac{\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = \frac{y_1 - y_2}{x_1 - x_2}, \qquad b = \frac{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

Substitution in the equation $y = mx + b$ gives the equation of the line determined by the given points.

(If $x_1 = x_2$, the line determined by the two points is the vertical line $x = x_1$.)

EXAMPLE 5–4. Find the equation of the line passing through the points (2, 5) and (6, 2). Check the results graphically (Fig. 5–3).

Substitution in the form $y = mx + b$ gives the two equations which determine m and b:

$$2m + b = 5, \qquad 6m + b = 2.$$

These equations are solved to find $-4m = 3$, and then by substitution, $b = 5 + 2 \cdot \frac{3}{4} = \frac{13}{2}$. Hence the equation is

$$y = -\tfrac{3}{4}x + \tfrac{13}{2}.$$

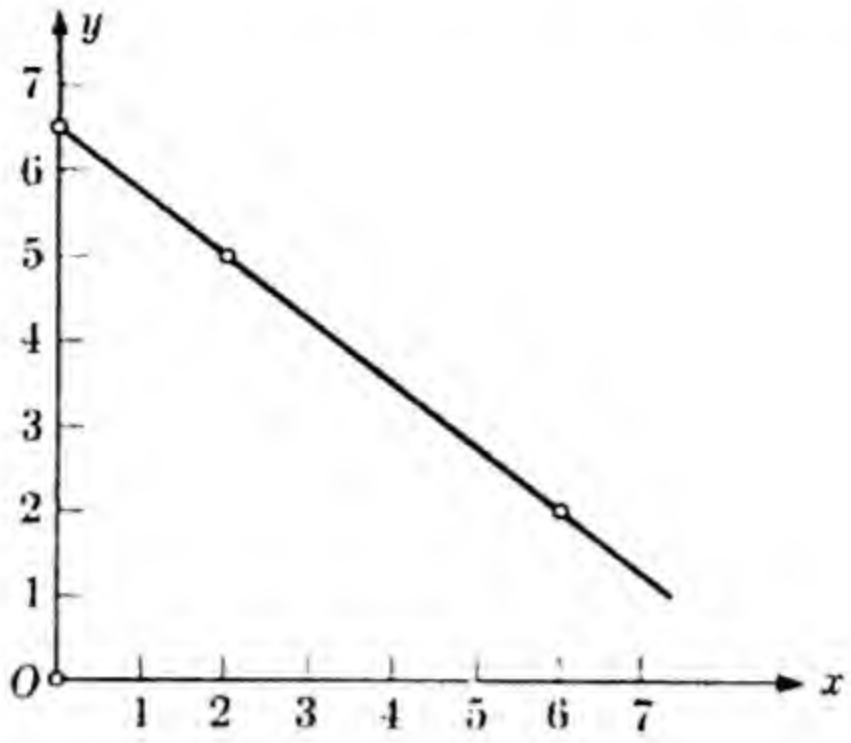


FIGURE 5–3

5–5 Worded problems and simultaneous linear equations. Many problems which involve two related, unknown quantities can be solved by means of simultaneous linear equations. Some problems are more conveniently solved in this way than by use of a single variable. The general procedure is to let x and y represent the two unknown quantities and from the problem determine two equations that relate x and y.

EXAMPLE 5–4 (see Example 4–6). The sum of two numbers is 24; one number is 3 more than twice the other. Find the numbers.

Represent the numbers by x and y. Then

$$x + y = 24, \qquad y = 2x + 3.$$

The elimination of y by substitution leads to the solution $x = 7$, $y = 17$.

EXAMPLE 5–5 (see Example 4–8). The sum of two numbers is 25 and the difference of their squares is 225. Find the numbers.

Represent the numbers by x and y. Then

$$x + y = 25, \qquad x^2 - y^2 = 225.$$

The elimination of y by substitution leads to an equation which reduces to a linear equation and then to the solution $x = 17$, $y = 8$.

Another procedure would be to divide in the form $(x^2 - y^2)/(x + y) = 225/25$ or $x - y = 9$. When this last equation is combined with $x + y = 25$, the same solutions are obtained.

EXAMPLE 5–6 (see Example 4–14). A man invests \$40,000 in government bonds yielding an annual return of $4\frac{1}{4}\%$ and in stocks yielding an annual return of 6%. The investments are made in units of \$100 each. He desires an annual income of \$2000. How many bonds and stocks should he buy?

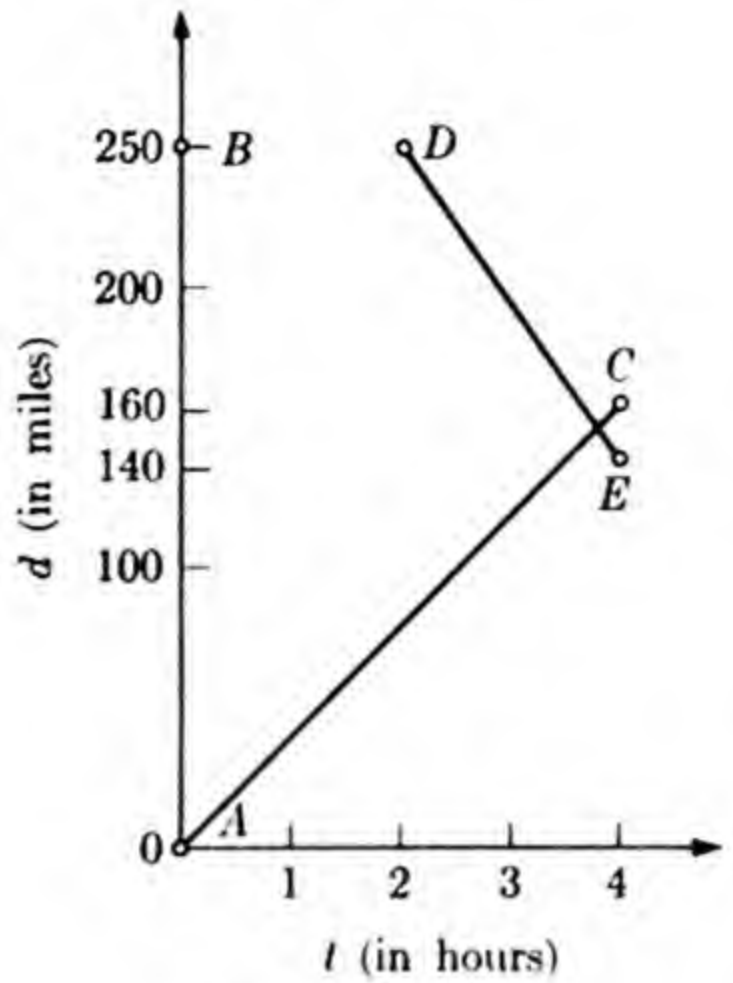


FIGURE 5–4

Let x be the number of \$100 bonds, and y be the number of \$100 stocks. Then the equations and their solutions are

$$\begin{array}{rl} x + y = 400 & 6 \\ \underline{4.25x + 6y = 2000} & -1 \\ 1.75x = 400, \quad x = 229, \quad y = 171, \text{ approx.} & \end{array}$$

EXAMPLE 5–7 (see Example 4–19). Two towns A and B are 250 mi apart. A car travels from A to B at the average rate of 40 mph. Two hours after the first car starts, a second car leaves B for A and travels at the average rate of 55 mph. When and where do they meet?

Let d be the distance in miles from A to where they meet, and let t be the time in hours that the first car travels. Figure 5–4 gives the graphical solution. The distance of the first car from A is represented by the line AC, where the coordinates of the points are $A(0, 0)$ and $C(4, 160)$. The distance of the second car from A is represented by the line DE, where the coordinates of the points are $D(2, 250)$, $E(4, 250 - 110 = 140)$. These two lines intersect approximately where $d = 150$ and $t = 3.8$.

The algebraic solution is found from the equations of these two lines.

$$AC\colon d = 40t; \qquad DE\colon d = 250 - 55(t - 2).$$

The simultaneous solution of these two linear equations yields the results: $t = 3\frac{5}{11}$ (hours) and $d = 151.6$ (miles).

EXAMPLE 5–8. Two boys sit on a teeter board supported at its center. They just balance when one sits 6 ft from the center and the other 8 ft from the center. If they exchange positions, it requires an additional

weight of 35 lb with the lighter boy to maintain the balance. How much does each boy weigh?

Let x and y be their weights in lb. The conditions of the problem give

$$6x = 8y \quad \text{and} \quad 8x = 6(y + 35).$$

Solving:

$$\begin{array}{rrr|r|r} 6x - 8y = 0 & & & -6 & -8 \\ \underline{8x - 6y = 210} & & & +8 & +6 \\ 28x & = 1680, & & x = 60, & \\ & 28y = 1260, & & y = 45. & \end{array}$$

EXAMPLE 5–9. How many quarts of milk containing 3% butterfat and of cream containing 18% butterfat should be combined to give 25 quarts of milk containing 5% butterfat?

Let x be the number of quarts of milk containing 3% butterfat, and let y be the number of quarts of cream containing 18% butterfat. Then

$$x + y = 25, \quad 3\%x + 18\%y = 5\%(25).$$

Multiply the second equation by 100 and the first by 3, and subtract:

$$\begin{array}{rrr} 3x + 3y = 75 & & \\ \underline{3x + 18y = 125} & & \\ 15y = 50, & y = \tfrac{10}{3}, & x = \tfrac{65}{3}. \end{array}$$

PROBLEM SET 5–2

1. Find the equation of the line which has the given slope m and passes through the given point P. Check your results graphically by use of the y-intercept.

(a) $m = \frac{3}{4}$, $P(-1, 3)$ (b) $m = -\frac{2}{3}$, $P(2, 4)$
(c) $m = \frac{2}{3}$, $P(-3, 5)$ (d) $m = -\frac{2}{5}$, $P(3, -5)$

2. Find the equation of the line which has the given y-intercept b and which passes through the given point P. Check your result graphically using the slope.

(a) $b = 3$, $P(4, 7)$ (b) $b = 3$, $P(2, -4)$
(c) $b = -\frac{2}{3}$, $P(-2, 3)$ (d) $b = -2$, $P(4, -2)$

3. Find the equation of the line which passes through the given points P and Q. Check your results graphically by use of the y-intercept. In problems (a), (b), (c) use the method of determinants; in problems (d), (e), (f) use another method.

(a) $P(2, 3)$, $Q(5, 8)$ (b) $P(3, 2)$, $Q(-1, 5)$
(c) $P(1, 1)$, $Q(-2, 4)$ (d) $P(4, 4)$, $Q(2, -3)$
(e) $P(-2, 3)$, $Q(4, -5)$ (f) $P(-3, -4)$, $Q(2, -1)$

Solve problems 4 to 20 by use of simultaneous linear equations in two variables.

4. The sum of two numbers is 10; their difference is half their sum. Find the numbers.

5. The sum of twice one number and the second is 12; the difference of the first and twice the second is 11. Find the numbers.

6. A line segment AB is $8\frac{1}{2}$ in. long. Locate the point C between A and B so that AC is $\frac{3}{2}$ inches longer than twice CB.

7. A line segment AB is 9 in. long. Locate the point C on this segment so that two-thirds of AC equals one-half of AB plus one-half of CB.

8. One part of \$3000 is invested at 5% and the remainder at 8%. The yearly income from both is \$186. How much is invested at each of these rates?

9. Mr. Rich wishes to invest \$$A$ in two types of investments, one which earns 4% with no risk, and the other that earns 16% with considerable risk. He would like to have an annual income of about 8% on his investment. (a) If he does not consider that any of his high yield investments will default, what part of his money should he invest in each? (b) If he considers that 5% of his high yield investment will default on both interest and principal, and he wishes his annual income to be 8% of his investment above the replacement of the defaulted principal, what part of his money should he invest in each?

10. The widow and a son are to receive \$2000 and \$1000, respectively, from an \$8000 estate. The rest is to be divided so that the widow receives $\frac{3}{2}$ as much of it as the son.

Let S be the son's total share, W the widow's total share. Show that $W + S = 8000$; $W - 2000 = \frac{3}{2}(S - 1000)$, and solve the problem graphically to determine the share of each.

11. The relation between the Centigrade and the Fahrenheit temperature scales is given by $F = \frac{9}{5}C + 32$. (a) When will the readings be the same? (b) When will the Fahrenheit reading be twice the Centigrade reading? (c) When will the Fahrenheit reading be 100 more than the Centigrade reading? Solve both algebraically and graphically (use one diagram).

12. Two towns A and B are 200 mi apart. A car travels from A to B at the average rate of 50 mph. One hour after the first car starts, a second car leaves B for A and travels at the average rate of 40 mph. When and where do they meet? Solve algebraically and graphically.

13. A freight train leaves Chicago for Denver and travels at the average rate of 35 mph. Three hours later a passenger train leaves Chicago for Denver and travels at an average rate of 60 mph. When and where will the passenger train overtake the freight? Solve algebraically and graphically.

14. Two boys sit on a teeter board supported at its center. They just balance when one sits 5 ft from the center and the other 7 ft from the center. If they exchange positions, it requires an additional weight of 25 lb with the lighter boy to maintain the balance. How much does each boy weigh?

15. Two men carry a long pole which carries a weight of 150 lb. One man holds the pole $4\frac{1}{2}$ ft from the weight and the other holds the pole 6 ft from the weight. How much of the weight does each lift?

16. A lever is in equilibrium when two unknown weights are placed $3\frac{1}{2}$ in. and $4\frac{1}{2}$ in., respectively, from its point of support. If an additional weight of 1 oz is

added to the lighter weight, it must be moved $\frac{1}{2}$ in. closer to the fulcrum to obtain equilibrium. What are the sizes of the weights?

17. How many gallons of milk containing 4% butterfat and of cream containing 15% butterfat should be combined to give 5 gallons of milk containing 6% butterfat?

18. How many pounds of candy worth \$.85 per lb and of candy worth \$.60 per lb should be mixed to obtain 50 lb of candy worth \$.75 per lb?

19. How many quarts of a mixture containing 45% alcohol should be combined with a mixture containing 70% alcohol to obtain 6 quarts of a mixture containing 55% alcohol.

20. A mixture of 5 lb of first grade coffee and 8 lb of second grade coffee is worth \$10. A mixture of 8 lb of the first grade coffee and 4 lb of the second grade coffee has the same value. What is each grade worth per pound?

5–6 Market equilibrium with linear demand and supply laws. Under pure competition where the price per unit quantity depends only on the quantity demanded and the supply available ("all other things being equal"), there is a tendency for the price to adjust itself so that the prices per unit quantity *demanded* and per unit quantity *supplied* are the same. If the price is too high, the consumer will not purchase, and if the price is too low, the supplier will not sell. *Market equilibrium* corresponds to the point where the demand curve and the supply curve (Section 4–7) intersect. The corresponding price p per unit quantity and the quantity x represent the *equilibrium price and quantity*. If the demand and supply laws are linear, the methods of Sections 5–2 and 5–3 can be used to find the equilibrium price and quantity. For a realistic situation, p and x should both be positive.

EXAMPLE 5–10. A demand law is given in the form $4x + 9p = 48$ and the supply law has the form $p = \frac{1}{9}x + 2$. Find the equilibrium price and quantity, first graphically and then algebraically.

The demand curve is drawn from the intercepts $(0, 16/3)$, $(12, 0)$. The point $(3, 4)$ serves as a check point. The supply curve is drawn from the points $(0, 2)$, $(9, 3)$. The two lines intersect at $x = 6$, $p = 2.7$, approx. (Fig. 5–5). The algebraic solution is obtained by substitution:

$$4x + 9(\tfrac{1}{9}x + 2) = 48$$

$$5x = 30$$

$$x = 6,$$

$$p = \tfrac{2}{3} + 2 = \tfrac{8}{3}.$$

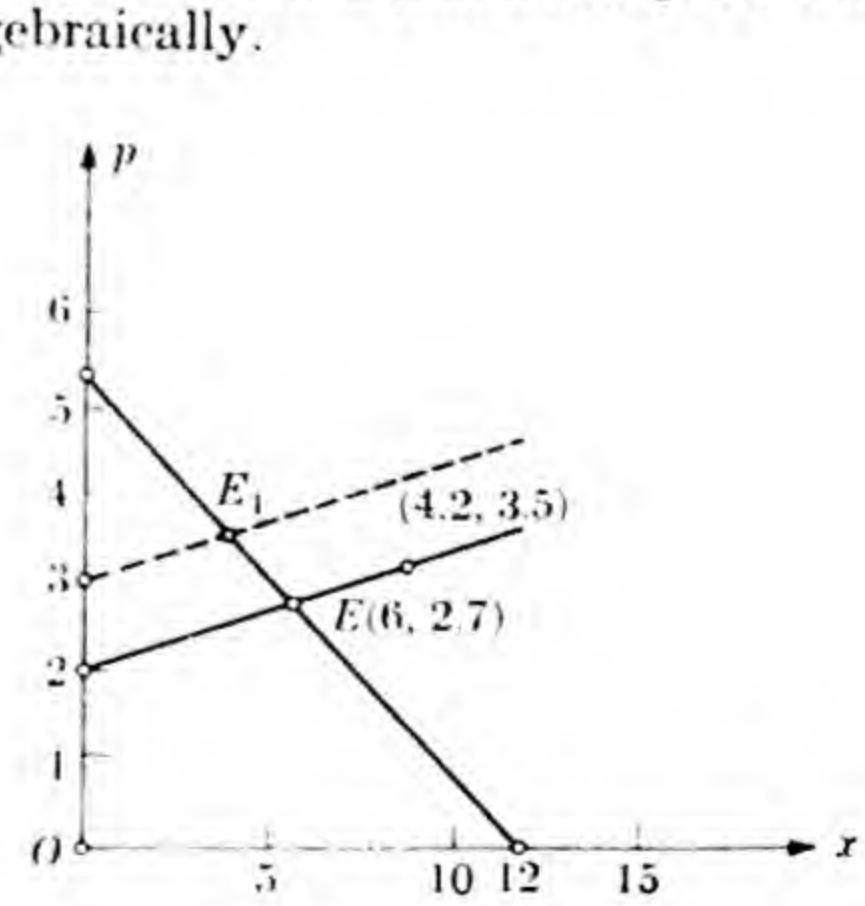


FIGURE 5–5

Additive tax and market equilibrium. If the government imposes a tax on a given commodity, the price to the consumer increases and the quantity demanded decreases. Consider the effect upon market equilibrium under pure competition and under the assumptions (1) that the quantity demanded by the consumer depends upon the price alone (that is, the demand function does not change), and (2) that if an additive tax of t per unit quantity is imposed, the producers adjust the supply to this new price.

If the supply law is given in the form $p = F(x)$ before taxation, it becomes $p_1 = F(x) + t$ after taxation, where p_1 is the new price per unit quantity. The new equilibrium point is found as the intersection of the original demand curve with the new supply curve. If the original supply curve is a straight line, the new supply curve is also a straight line, parallel to the original line, a vertical distance equal to t above it.

If the original supply law is given in a form different from $p = F(x)$, it may be possible to obtain this form by solving for p. The supply law after taxation is then written by the addition of t.

A subsidy may be considered as a negative tax. The supply curve is moved downward by an amount equal to the subsidy. The price to the consumer decreases by an amount to be determined by comparing the old and new equilibrium conditions. The quantity demanded increases.

EXAMPLE 5–11. Using the demand and supply laws of Example 5–10:

$$4x + 9p = 48, \qquad p = \tfrac{1}{9}x + 2,$$

suppose an additive tax of 1 per unit quantity is imposed. Find the new equilibrium conditions.

The new equilibrium conditions are obtained from the equations

$$4x_1 + 9p_1 = 48,$$

$$p_1 = \tfrac{1}{9}x_1 + 2 + 1,$$

where the subscripts indicate the new conditions. The new supply line is shown dotted in Fig. 5–5. The algebraic solution gives

$$4x_1 + 9(\tfrac{1}{9}x_1 + 3) = 48$$

$$5x_1 = 21$$

$$x_1 = 4.2, \qquad p_1 = 3 + \tfrac{1}{9} \cdot \tfrac{21}{5} = 3.47, \text{ approx.}$$

These results are consistent with the diagram. Comparison with the results of Example 5–10 gives the price *increase* as $p_1 - p = 3.47 - 2.67 = 0.8$ and the quantity demanded decrease as $x - x_1 = 6 - 4.2 = 1.8$.

PROBLEM SET 5–3

1. (a) If the demand law is $p = 12 - 3x$ and the supply law is $p = \frac{3}{2}x + 2$, find the equilibrium price and quantity. Check graphically. Do not use the equilibrium price and quantity found to draw either line.

(b) If a tax of 1 per unit quantity is imposed on the commodity, find the new equilibrium price and quantity, the increase in price and the decrease in quantity demanded. Check graphically. Use the diagram already constructed in part (a).

2. With the demand and supply laws of problem 1, (a) show that if an additive tax of t is imposed on each unit of quantity, then the increase in price will be $\frac{2}{3}t$. (b) What additive tax would increase the price by $\frac{3}{2}$?

3. (a) If the demand law is $p = 36 - 5x$ and the supply law is $p = 9 + 4x$, find the equilibrium price and quantity. Check graphically; do not use the equilibrium point to draw either line.

(b) If a subsidy of 4 per unit quantity is granted to the producer, find the new equilibrium price and quantity, the decrease in price and the increase in quantity demanded. Check graphically. Use the diagram already constructed in part (a).

4. With the demand and supply laws of problem 3(a), show that the decrease in price is a constant multiple of the subsidy.

5. (a) The demand and supply laws are given in the form

$$x = 15 - \tfrac{3}{2}p, \qquad x = 3p - 6.$$

Find the equilibrium price and quantity. Check graphically.

(b) If a tax of 2 per unit quantity is imposed upon the commodity, find the new equilibrium conditions. Check graphically.

6. (a) If the demand and supply laws are given in the form of the equations of problem 5(a), and an additive tax of t per unit is imposed, find the new equilibrium price and quantity as functions of t. Check the answers against the results in 5(a) and 5(b).

(b) If a subsidy of $s = 2$ is granted, what is the decrease in price and what is the increase in the quantity demanded?

7. The demand and supply laws are given in the form

$$x = 10 - 2p, \qquad x = 6p - 6.$$

What additive tax will cause the price to increase by $1\frac{1}{2}$ units? Solve the problem graphically first and then algebraically.

8. (a) The demand and supply laws are given in the form

$$2x + 3p = 18, \qquad 6p - 3x = 8.$$

Find the equilibrium price and quantity. Check graphically.

(b) If a subsidy of 4/3 price units is granted per unit quantity, determine the new equilibrium conditions. Check graphically using the diagram constructed in part (a).

9. If the demand law is $12x + 8p = 27$ and the supply law is $p = \frac{2}{3}x + \frac{2}{3}$, find the equilibrium price and quantity before and after a tax $t = \frac{1}{2}$ is imposed. Check graphically.

10. If the demand and supply laws are

$$12x + 8p = 27, \qquad p = \tfrac{2}{3}x + \tfrac{2}{3},$$

and a tax of t per unit quantity is imposed, find the new equilibrium price and show that the increase in price is $\frac{9}{13}t$.

5–7 Determinants of the third order. In order to formalize and simplify procedures for finding the solution of three simultaneous linear equations in three variables, determinants of the third order are introduced. A *determinant* of the third order is a square array of nine elements written within a pair of vertical bars,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

to which is assigned a value in the manner discussed below.

The *minor* of any element is the second-order determinant obtained by striking out the row and column which contain that element. Thus the minor of b_1 is the determinant

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = a_2c_3 - c_2a_3.$$

The *cofactor* of any element is its minor together with its sign of position. The sign of position is obtained by assigning the + sign to the upper left-hand element and then alternating the + and − signs throughout the pattern. The sign of position is positive if the sum of the row and column in which the element is found is even. The sign of position is negative if the sum of the row and column in which the element is found is odd. The cofactor of any element is represented by the corresponding capital letter with the same subscript. For example, b_1 is in the first row and second column, its sign of position is negative and its cofactor is

$$B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

The value, D, assigned to the third-order determinant is

$$D = a_1A_1 + a_2A_2 + a_3A_3. \tag{5–12}$$

In expanded form, this becomes

$$D = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1. \qquad (5\text{–}13)$$

This value is described by expanding the determinant in terms of the elements of the first column. It is not difficult to show that the value of the determinant may be found by expanding in terms of the elements of any column or any row. Thus

$$D = b_1B_1 + b_2B_2 + b_3B_3, \qquad (5\text{–}14)$$

$$D = a_1A_1 + b_1B_1 + c_1C_1, \qquad (5\text{–}15)$$

with similar expansions involving the third column, the second row, or the third row. To verify Eqs. (5–14) and (5–15), it is only necessary to regroup the six terms of Eq. (5–13). For Eq. (5–14), regroup as

$$D = -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$
$$= b_1B_1 + b_2B_2 + b_3B_3.$$

For Eq. (5–15), regroup as

$$D = a_1(b_2c_3 - b_2c_3) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$
$$= a_1A_1 + b_1B_1 + c_1C_1.$$

A comparison of Eqs. (5–12) and (5–15) shows that *the value of a determinant is not changed if rows and columns are interchanged.* Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad (5\text{–}16)$$

To find the value of a determinant, it is often convenient to replace the given determinant by one which has the same value but for which all except one of the elements in a column (or row) are zero. Expansion in terms of the elements of this column replaces the third-order determinant by a second-order determinant. To develop this procedure, the following properties of determinants are needed:

(A) *The value of a determinant is zero if the elements of two columns are the same or are proportional.*

(B) *The value of the determinant is unchanged if the elements of any column are multiplied by the same number and added to the corresponding elements of any other column.*

From these two properties a third one is proved:

(C) *If two columns are interchanged, the value of the determinant changes sign.*

Property (A) is proved for the case where $c_1 = kb_1, c_2 = kb_2, c_3 = kb_3$, that is,

$$D = \begin{vmatrix} a_1 & b_1 & kb_1 \\ a_2 & b_2 & kb_2 \\ a_3 & b_3 & kb_3 \end{vmatrix} = 0. \tag{5–17}$$

If this determinant is expanded in terms of the elements of the first column:

$$A_1 = k(b_2b_3 - b_2b_3) = 0 = A_2 = A_3,$$

so that

$$D = 0.$$

The same result could be obtained from Eq. (5–13) by replacing c_1, c_2, c_3, by the values given above. If the same determinant is expanded in terms of the elements of the third column, the following formula is obtained:

$$b_1C_1 + b_2C_2 + b_3C_3 = 0.$$

There are five similar formulas, one of which is

$$a_1B_1 + a_2B_2 + a_3B_3 = 0. \tag{5–18}$$

Property (B) is proved for the case

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D' = \begin{vmatrix} a_1 & b_1 + ka_1 & c_1 \\ a_2 & b_2 + ka_2 & c_2 \\ a_3 & b_3 + ka_3 & c_3 \end{vmatrix}. \tag{5–19}$$

Expand in terms of the elements of the second column and use Eqs. (5–14) and (5–18).

$$\begin{aligned} D' &= (b_1 + ka_1)B_1 + (b_2 + ka_2)B_2 + (b_3 + ka_3)B_3 \\ &= (b_1B_1 + b_2B_2 + b_3B_3) + k(a_1B_1 + a_2B_2 + a_3B_3) \\ &= D + 0 = D. \end{aligned}$$

Property (C) is proved for the case where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D' = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}.$$

Apply property (B) and expand:

$$D' = \begin{vmatrix} b_1 & a_1 - b_1 & c_1 \\ b_2 & a_2 - b_2 & c_2 \\ b_3 & a_3 - b_3 & c_3 \end{vmatrix} = (a_1 - b_1)B_1 + (a_2 - b_2)B_2 + (a_3 - b_3)B_3$$
$$= (a_1B_1 + a_2B_2 + a_3B_3) - (b_1B_1 + b_2B_2 + b_3B_3)$$
$$= 0 - D = -D.$$

Proofs for other cases of (A), (B), (C) and for cases where "column" is replaced by "row" in the rules are similar to the foregoing.

EXAMPLE 5–12. Find the value of

$$D = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & -3 \\ 3 & 4 & 1 \end{vmatrix}.$$

(a) Expand in terms of the elements of the first column:

$$D = 1(5 + 12) - 2(1 - 8) + 3(-3 - 10)$$
$$= 17 + 14 - 39 = -8.$$

(b) Expand in terms of the elements of the first row:

$$D = 1(5 + 12) - 1(2 + 9) + 2(8 - 15)$$
$$= 17 - 11 - 14 = -8.$$

(c) If we subtract twice the elements of the first row from the elements of the second row, and then subtract three times the elements of the first row from the elements of the third row, we find

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & -3 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -7 \\ 0 & 1 & -5 \end{vmatrix} = 1(-15 + 7) = -8.$$

(d) We could proceed in various ways to obtain zeros in other positions. For example, leaving the third row unchanged, we find

$$\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -7 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 7 \\ 0 & 0 & 8 \\ 0 & 1 & -5 \end{vmatrix},$$

and by further combinations and interchanging two columns, we obtain

$$D = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{vmatrix} = -8.$$

5–8 Simultaneous linear equations in three variables. The remarks made in Section 5–1 concerning systems of equivalent equations apply to three simultaneous equations in three variables. The algebraic methods of Section 5–2 can be adapted to such systems. These may be written in the form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{5–20}$$

(A) *Solution by elimination.* By multiplying the first of these equations by a_2, the second by $-a_1$ and adding, the variable x is eliminated; in an analogous manner, x can be eliminated in a combination of the first and third of these equations. This gives two equations,

$$b_4y + c_4z = d_4, \qquad b_5y + c_5z = d_5, \tag{5–21}$$

which, with the first of the equations in the set (5–20), form a set which is equivalent to the original set. By eliminating y between these two equations, we obtain an equation of the form

$$c_6z = d_6. \tag{5–22}$$

If $c_6 \neq 0$, find z from Eq. (5–22), then by substitution find y from Eq. (5–21), and then find x from Eq. (5–20). It is important to make the eliminations systematically, but which variable is first eliminated is non-essential. In some cases it may be simpler to eliminate z first and then x to find y, and then complete the solution by substitution.

EXAMPLE 5–13. Solve the system of equations

$$\begin{array}{rrrrl|l} 3x & -2y & +4z & = 3 & (1) & 1 \\ 4x & +3y & & = 9 & (2) & \\ 2x & +4y & +z & = 0 & (3) & -4 \end{array}.$$

Since equation (2) does not contain z, eliminate z between equations (1) and (3) and combine this with equation (2):

$$\begin{array}{rrrrl|l} 4x & +3y & = 9 & (2) & 6 \\ -5x & -18y & = 3 & (4) & 1 \\ \hline 19x & & = 57; & x = 3. & \end{array}$$

Then

$$12 + 3y = 9 \qquad \text{gives} \qquad y = -1,$$

and

$$9 + 2 + 4z = 3 \qquad \text{gives} \qquad z = -2.$$

The solution $(3, -1, -2)$ is checked in equation (3): $6 - 4 - 2 = 0$.

(B) *Solution by determinants.* The determinant formed from the coefficients is

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{5-23}$$

and the cofactor of each element is represented by the corresponding capital letter. The procedure for solving the equations by determinants is shown in the following form:

$$\begin{array}{l|l|l|l} a_1x + b_1y + c_1z = d_1 & A_1 & B_1 & C_1 \\ a_2x + b_2y + c_2z = d_2 & A_2 & B_2 & C_2 \\ \underline{a_3x + b_3y + c_3z = d_3} & A_3 & B_3 & C_3 \end{array}$$

$$\begin{aligned} Dx &= D_1 \\ Dy &= D_2 \\ Dz &= D_3, \end{aligned} \tag{5-24}$$

where

$$\begin{aligned} a_1A_1 + a_2A_2 + a_3A_3 &= D, \\ a_1B_1 + a_2B_2 + a_3B_3 &= 0, \\ a_1C_1 + a_2C_2 + a_3C_3 &= 0, \end{aligned}$$

have been used to find x. Similar relations are used to find y and z. The expressions for D_1, D_2, and D_3 are:

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}. \tag{5-25}$$

Provided the equations are written in the precise form of (5–20), D_1, D_2, and D, are obtained from D by replacing the a's, b's, and c's, respectively, by the d's. The determination of the actual cofactors is not required to write the set of Eqs. (5–24) and (5–25).

Cramer's Rule. If $D \neq 0$, Eqs. (5–24) give

$$x = \frac{D_1}{D}, \qquad y = \frac{D_2}{D}, \qquad z = \frac{D_3}{D} \tag{5-26}$$

as the unique solution of the system (5–20), where D, D_1, D_2, D_3 are defined by Eqs. (5–23) and (5–25). The four determinants may be evaluated by any convenient method.

EXAMPLE 5–14. Solve the following system of equations by means of determinants.

$$\begin{aligned} x - 2y - 3z &= 2 \\ -2x + 3y + z &= -5 \\ 3x - 4y - 2z &= 7. \end{aligned}$$

$$D = \begin{vmatrix} 1 & -2 & -3 \\ -2 & 3 & 1 \\ 3 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \\ 0 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \\ 0 & 0 & -3 \end{vmatrix} = 3,$$

$$D_1 = \begin{vmatrix} 2 & -2 & -3 \\ -5 & 3 & 1 \\ 7 & -4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -3 \\ -1 & -1 & -5 \\ 1 & 2 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 7 \\ -1 & -1 & -5 \\ -1 & 0 & -3 \end{vmatrix} = -1(-12 + 7) = 5,$$

$$D_2 = \begin{vmatrix} 1 & 2 & -3 \\ -2 & -5 & 1 \\ 3 & 7 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -1 & -5 \\ 0 & 1 & 7 \end{vmatrix} = -7 + 5 = -2,$$

$$D_3 = \begin{vmatrix} 1 & -2 & 2 \\ -2 & 3 & -5 \\ 3 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -1 + 2 = 1.$$

Therefore

$$x = \frac{5}{3}, \qquad y = \frac{-2}{3}, \qquad z = \frac{1}{3}.$$

A check is obtained by substitution:

$$3\left(\frac{5}{3}\right) - 4\left(\frac{-2}{3}\right) - 2\left(\frac{1}{3}\right) = \frac{15 + 8 - 2}{3} = \frac{21}{3} = 7.$$

PROBLEM SET 5–4

1. Evaluate the determinant

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 3 \\ 3 & 10 & 8 \end{vmatrix}$$

by each of the following methods:

(a) expand in terms of the elements of the first column;
(b) expand in terms of the elements of the first row;
(c) expand in terms of the elements of the second row.

2. Proceed as in (a), (b), (c) of problem 1 for

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 5 \\ 3 & 10 & 8 \end{vmatrix}.$$

3. Proceed as in (a), (b), (c) of problem 1 for

$$\begin{vmatrix} 2 & 3 & 4 \\ -3 & 4 & 2 \\ 4 & 5 & 6 \end{vmatrix}.$$

4. Proceed as in (a), (b), (c) of problem 1 for

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}.$$

5. Proceed as in (a), (b), (c) of problem 1 for

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 3 & 1 & 2 \\ 5 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}.$$

6. Evaluate each of the following determinants by combining rows (or columns) to obtain two zeros in some column (or row) and then reducing the third-order determinant to one second-order determinant.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 3 \\ 3 & 10 & 8 \end{vmatrix}$, (b) $\begin{vmatrix} 2 & 3 & 4 \\ -3 & 4 & 2 \\ 4 & 5 & 6 \end{vmatrix}$, (c) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$.

7. By a combination of rows, reduce the determinant

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 5 \\ 3 & 10 & 8 \end{vmatrix}$$

to an equivalent one where all elements except those in the "principal diagonal" (see Example 5–12(d)) are zero.

8. Proceed as in problem 7 for

$$\begin{vmatrix} 3 & 1 & 2 \\ 5 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix},$$

except that you may combine either rows or columns.

9. By regrouping terms of Eq. (5–13), show that

(a) $D = c_1C_1 + c_2C_2 + c_3C_3$,

(b) $D = a_3A_3 + b_3B_3 + c_3C_3$.

10. By regrouping terms of Eq. (5–13), or by using other properties of determinants, show that

(a) $a_1B_1 + a_2B_2 + a_3B_3 = 0$,

(b) $a_1A_3 + b_1B_3 + c_1C_3 = 0$.

Write other similar equations.

11. Solve the following sets of linear equations by the method of successive elimination. Include a check.

(a)
$$\begin{aligned} x - 2y - 3z &= 2 \\ -2x + 3y + z &= -5 \\ 3x - 4y - 2z &= 7 \end{aligned}$$

(b)
$$\begin{aligned} x + 3y + 4z &= 1 \\ 2x + 7y + 3z &= -5 \\ 3x + 10y + 8z &= -3 \end{aligned}$$

(c)
$$\begin{aligned} -x + y + z &= 2 \\ x - y + z &= 3 \\ x + y - z &= 4 \end{aligned}$$

(d)
$$\begin{aligned} -3x + 2y + z &= 2 \\ x + 3y - 2z &= 4 \\ -4x + 5y + 3z &= 6 \end{aligned}$$

12. Solve by means of determinants.

$$\begin{aligned} 3x - 2y + 4z &= 3 \\ 4x + 3y &= 9 \\ 2x + 4y + z &= 0 \end{aligned}$$

(See Example 5–13).

13. Solve each of the problems 11(a), (b), (c), (d) by means of determinants. Use several different procedures for evaluating the determinants. Include a check by substitution.

5–9 Graphical interpretation. Consider three mutually perpendicular lines (drawn in perspective) meeting at O (Fig. 5–6). The point O is called

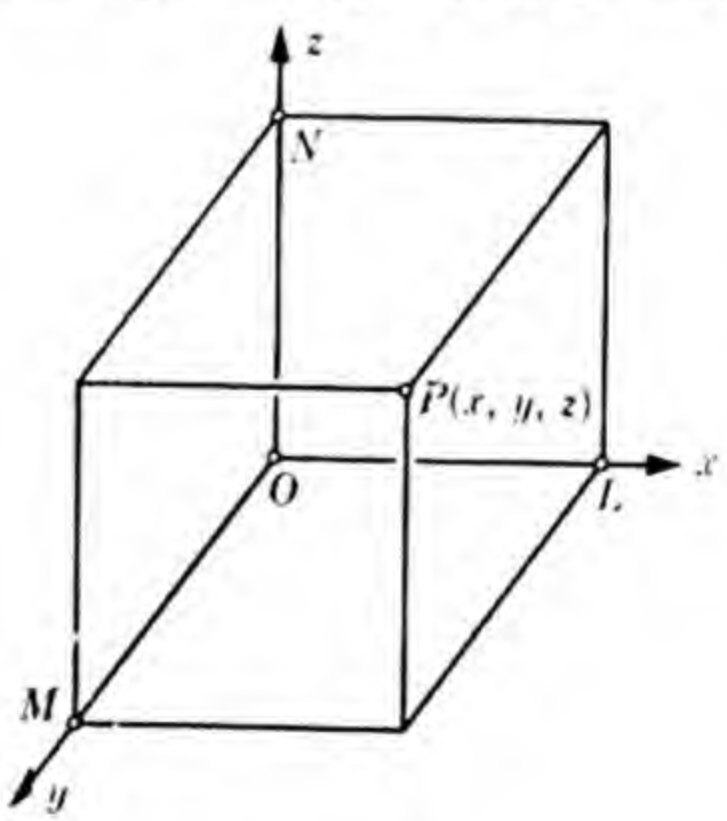


FIGURE 5-6

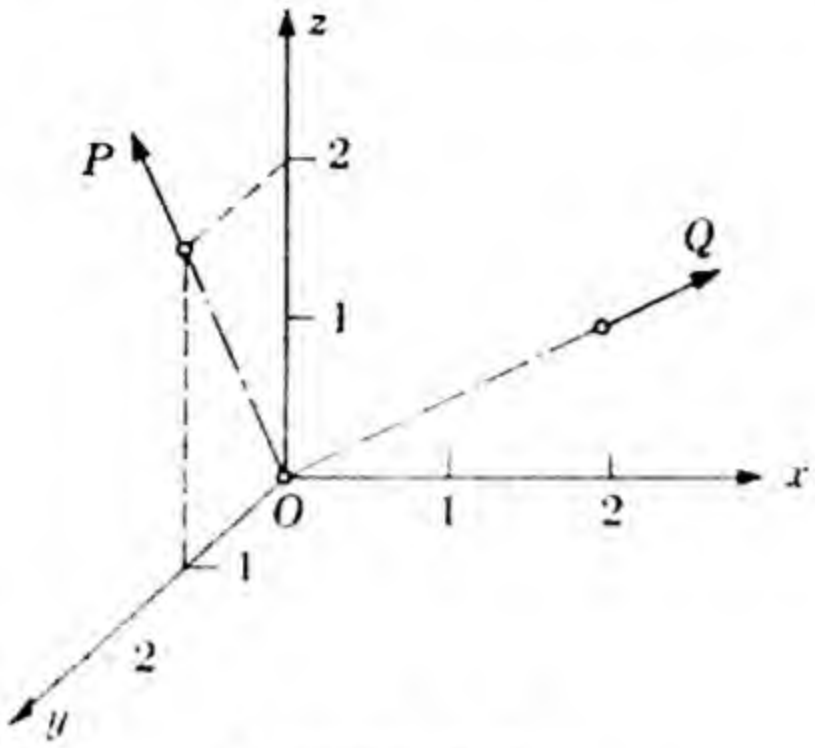


FIGURE 5–7

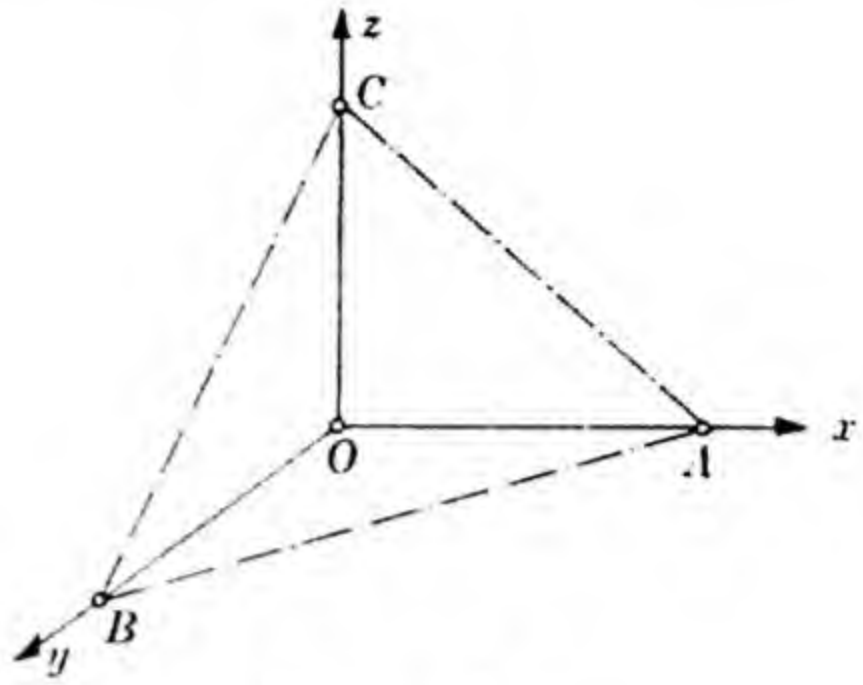


FIGURE 5–8

the *origin* and the lines OX, OY, OZ are called the *coordinate axes.* The planes YOZ, ZOX, XOY are called the *coordinate planes.* Let P be any point in space. Through P draw planes parallel to the coordinate planes intersecting the coordinate axes in L, M, N, respectively. On these axes let the directed distances OL, OM, ON be x, y, z, respectively. To each point P corresponds a unique ordered number triple (x, y, z). Conversely, given any such ordered number triple, reversal of the construction locates a unique point P. In this way a 1–1 correspondence between points in space and number triples is obtained. This correspondence is represented by $P(x, y, z)$. From the construction, it is seen that the coordinate planes are characterized by the equations $x = 0$, $y = 0$, $z = 0$ and the coordinate axes are represented by taking these equations in pairs. The plane YOZ has the equation $x = 0$, and the x-axis, OX, has the equations $y = 0$, $z = 0$.

It is assumed without proof* that the graph of a linear equation

$$ax + by + cz = d$$

is a plane in space. To make a perspective diagram of the general plane, distinguish between the cases when the plane passes through the origin, $d = 0$ (Fig. 5–7), and when it does not pass through the origin, $d \neq 0$ (Fig. 5–8). In the former case, find the traces of the given plane on two of the coordinate planes, for example, the plane $x = 0$, with trace OP, and the planes $y = 0$ with trace OQ. The equations of these traces are

$$OP: x = 0, \qquad by + cz = 0;$$
$$OQ: y = 0, \qquad ax + cz = 0.$$

The lines in these coordinate planes can be drawn in perspective in the diagram, and the required plane is determined by the lines OP and OQ.

* The proof is given in Appendix I.

If the plane does not pass through O, draw the three traces

$$\begin{aligned} BC{:}\, x &= 0, \quad & by + cz &= d; \\ CA{:}\, y &= 0, \quad & ax + cz &= d; \\ AB{:}\, z &= 0, \quad & ax + by &= d; \end{aligned}$$

by first finding that the given plane cuts the coordinates axes at the points $A(d/a, 0, 0)$, $B(0, d/b, 0)$, $C(0, 0, d/c)$. The required plane is then determined by the three points A, B, C.

EXAMPLE 5–15. Sketch the plane $x + 4y - 2z = 0$.

In the $x = 0$ plane, $y = \frac{1}{2}z$, and this line can be drawn through the origin and through the point $(0, 1, 2)$. In the $y = 0$ plane, $x = 2z$, and this line can be drawn through the origin and the point $(2, 0, 1)$. These lines are shown in Fig. 5–7. The trace in the $z = 0$ plane, $x + 4y = 0$, would be more difficult to draw.

EXAMPLE 5–16. A total expenditure of 20 units is made for three commodities where the prices are 3, 4, 5 units, and the corresponding quantities are x, y, and z. The budget equation is $3x + 4y + 5z = 20$. Sketch the corresponding budget plane.

The three intercepts are found to be $A(20/3, 0, 0)$, $B(0, 5, 0)$, $C(0, 0, 4)$ and from these three points we can picture that part of the plane that has significance $(x, y, z \geqq 0)$.

Under certain competitive conditions, the sale of one commodity influences favorably the sale of a related commodity and the quantity x demanded by the consumer of the one commodity depends upon the price p of that commodity and also the price q of the related commodity. This dependence is such that x increases as p decreases and also increases as q decreases. The simplest such demand law arises when x is a linear function of p and q with the coefficients of p and q both negative.

EXAMPLE 5–17. The demand law is given in the form $x = 12 - 2p - q$, where p is the price of each unit of the commodity, x is the quantity demanded, and q is the price of each unit of the "complementary" commodity. Sketch the corresponding plane, taking the x- and p-axes as in Section 4–7 and the q-axis as the third axis in space.

Note that x decreases if either p or q increases and x increases if either p or q decreases. It is desirable to select the p-unit equal to the q-unit, but the diagram must be viewed in perspective. The x-unit is chosen for convenience. The equation can be written in the convenient form

$$\frac{x}{12} + \frac{q}{12} + \frac{p}{6} = 1,$$

from which the intercepts can be read: $OA = 12$, $OB = 12$, $OC = 6$.

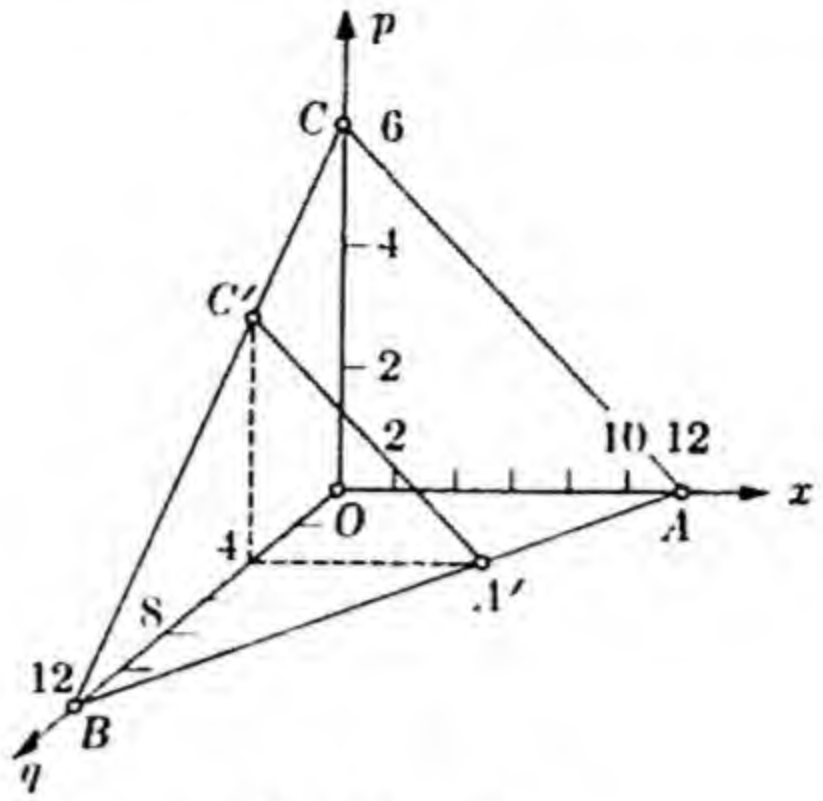


FIGURE 5–9

The diagram (Fig. 5–9) also shows the demand line $A'C'$ corresponding to fixed $q = 4$ and variable x and p. The equations of this line are $q = 4$, $x = 8 - 2p$.

Inconsistent and dependent equations. As shown earlier, the solution of three simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z_1 &= d_1, \\ a_2x + b_2y + c_2z_2 &= d_2, \\ a_3x + b_3y + c_3z_3 &= d_3, \end{aligned} \tag{5–20}$$

by means of determinants uses the equations:

$$Dx = D_1, \qquad Dy = D_2, \qquad Dz = D_3. \tag{5–24}$$

(1) If $D \neq 0$, they can be solved for the unique solution.

In general, the three planes which correspond to the three equations in x, y, z have one and only one point in common and this point is found algebraically as

$$x = \frac{D_1}{D}, \qquad y = \frac{D_2}{D}, \qquad z = \frac{D_3}{D},$$

where D, D_1, D_2, D_3 are given in Eqs. (5–23) and (5–25).

(2) *If* $D = 0$, *and* D_1, D_2, D_3 *are not all zero, Eqs.* (5–20) *are inconsistent, that is, have no solution.*

The assumption that Eqs. (5–20) have a solution under conditions (2) leads to the contradiction $0 \cdot x \neq 0$. There are two possible geometric situations where this may arise. (a) If two of the planes are parallel, they have no common point and the corresponding equations can have no set of values for x, y, z which simultaneously satisfy both. This case arises

when two of the equations can be reduced to the form

$$ax + by + cz = d_1 \qquad \text{and} \qquad ax + by + cz = d_2, \qquad d_1 \neq d_2.$$

(See proof in Appendix I.) (b) If no two of the planes are parallel but the line of intersection of two of the planes is parallel to the third plane, the three planes have no common point. This case arises when $D = 0$, no two of the planes are parallel, and one of D_1, D_2, D_3 is different from 0.

(3) If $D = D_1 = D_2 = D_3 = 0$, the equations are either inconsistent or dependent. If the three corresponding planes are parallel, the equations have no common solution. If the three corresponding planes are not parallel, they have a line in common and the equations are *dependent*. One of the variables, say z, could be assigned arbitrarily and the values of x and y determined from the system

$$a_1x + b_1y = (d_1 - c_1z); \qquad a_2x + b_2y = d_2 - c_2z.$$

(The case when two of the planes are identical needs no special discussion.)

EXAMPLE 5–18. Show that the following systems of equations are inconsistent, and discuss the corresponding geometric situations.

(a)
$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 4y + 6z &= 8 \\ 3x - 2y + z &= 4 \end{aligned}$$

(b)
$$\begin{aligned} 3x - 2y &= 5 \\ 2x - z &= 4 \\ x + 2y - 2z &= 6 \end{aligned}$$

In the set (a) it is recognized that the first and second equations give parallel planes $x + 2y + 3z = 6$, $x + 2y + 3z = 4$. If the method of eliminating x between the first two equations is applied, $0 \cdot x + 0 \cdot y + 0 \cdot z = -4$, which is impossible.

In the set (b), it is recognized that no two of the corresponding planes are parallel. However

$$D = \begin{vmatrix} 3 & -2 & 0 \\ 2 & 0 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 0 \\ 2 & 0 & -1 \\ -3 & 2 & 0 \end{vmatrix} = 0,$$

$$D_1 = \begin{vmatrix} 5 & -2 & 0 \\ 4 & 0 & -1 \\ 6 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 5 & -2 & 0 \\ 4 & 0 & -1 \\ -2 & 2 & 0 \end{vmatrix} \neq 0.$$

It is not necessary to compute D_2 and D_3. The three planes are such that the line of intersection of two of them is parallel to the third.

EXAMPLE 5-19. Show that the following systems of equations are dependent, that is, have an infinite number of solutions. Discuss the corresponding geometric situations.

(a) $\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 4y + 6z &= 12 \\ 3x - 2y + z &= 2 \end{aligned}$ (b) $\begin{aligned} 3x - 2y &= 5 \\ 2x - z &= 4 \\ 4y - 3z &= 2 \end{aligned}$

In the set (a) the first and second equations represent the same plane, but this plane is neither parallel nor identical with the plane that corresponds to the third equation. To find the complete solution, consider that z may have any arbitrary value and solve the system

$$\begin{aligned} x + 2y &= 6 - 3z \\ 3x - 2y &= 2 - z \\ \hline 4x &= 8 - 4z \\ x &= 2 - z; \\ 2y &= 6 - 3z - 2 + z = 4 - 2z. \end{aligned}$$

Hence the most general solution is $(2 - z, 2 - z, z)$. The line could be drawn by plotting two points, say $(2, 2, 0)$ and $(0, 0, 2)$.

In the set (b) no two of the planes are parallel or identical. However

$$D = \begin{vmatrix} 3 & -2 & 0 \\ 2 & 0 & -1 \\ 0 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 0 \\ 2 & 0 & -1 \\ -6 & 4 & 0 \end{vmatrix} = 0,$$

$$D_1 = \begin{vmatrix} 5 & -2 & 0 \\ 4 & 0 & -1 \\ 2 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 5 & -2 & 0 \\ 4 & 0 & -1 \\ -10 & 4 & 0 \end{vmatrix} = 0,$$

$$D_2 = \begin{vmatrix} 3 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 0 \\ 2 & 4 & -1 \\ -6 & -10 & 0 \end{vmatrix} = 0,$$

$$D_3 = \begin{vmatrix} 3 & -2 & 5 \\ 2 & 0 & 4 \\ 0 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 5 \\ 2 & 0 & 4 \\ 6 & 0 & 12 \end{vmatrix} = 0.$$

Hence the equations are dependent, representing three planes through the same line. From the last two equations, we obtain

$$x = \tfrac{1}{2}(z + 4), \qquad y = \tfrac{1}{4}(3z + 2).$$

The general solution is

$$\left(\frac{z + 4}{2}, \frac{3z + 2}{4}, z\right).$$

One of the most important cases of dependent equations occurs when $d_1 = d_2 = d_3 = 0$, in which case the equations are called *homogeneous linear equations*:

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0. \end{aligned} \tag{5–27}$$

The origin $O(0, 0, 0)$ is a common solution. If the equations represent distinct planes, they have no other solution unless $D = 0$. It is noted that $D_1 = D_2 = D_3 = 0$, since one column of each is a column of zeros. If $D \neq 0$, the planes have only the origin in common, but if $D = 0$, the planes have a line in common.

EXAMPLE 5–20. Show that the equations

$$x - 2y = 0, \qquad 3x - 4z = 0, \qquad x + 4y - 4z = 0$$

are dependent, and illustrate the situation graphically. Find their common line by means of the origin and some other convenient point. Use this line and the traces of these planes in the $z = 0$, $y = 0$, $x = 0$ planes, respectively.

The determinant of the coefficients is

$$D = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 0 & -4 \\ 1 & 4 & -4 \end{vmatrix}$$

$$= 1(16) + 2(-12 + 4) = 0.$$

Since the planes are distinct, they have a line in common. The first two equations give

$$y = \frac{x}{2}, \qquad z = \frac{3}{4}x.$$

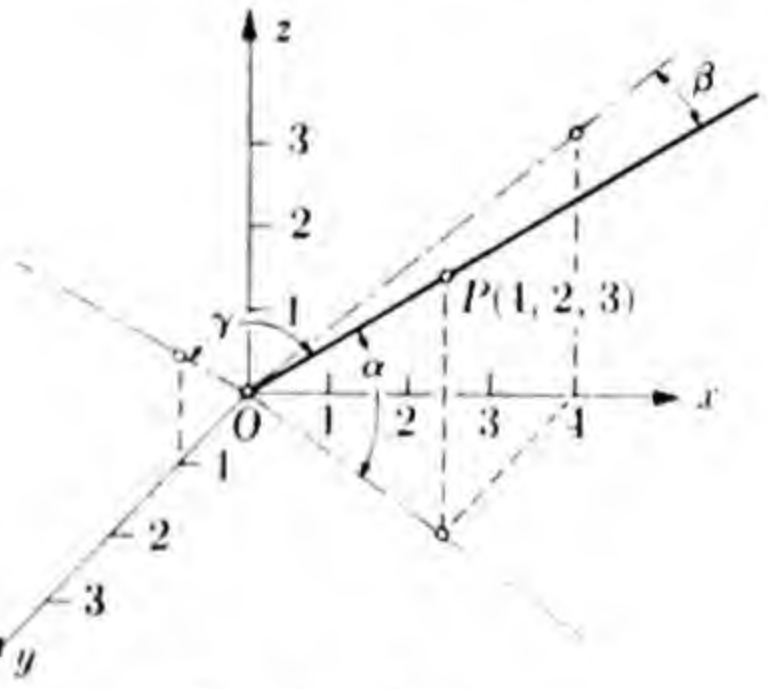


FIGURE 5-10

If we take $x = 4$, we find the convenient point $P(4, 2, 3)$. The trace of the first plane in the $z = 0$ plane is $x - 2y = 0$. This line and the line OP determines the plane α. The trace of the second plane in the $y = 0$ plane is $3x - 4z = 0$ and so the plane β can be visualized. The trace of the third plane in the $x = 0$ plane is $y - z = 0$ and so this plane γ can also be visualized (Fig. 5–10).

A problem closely related to the one above but which has an entirely different geometric interpretation is that of determining whether three distinct, nonparallel lines in the xy-plane are linearly dependent. In general, three such lines form a triangle, but if they all pass through the same

point, they are *linearly dependent.* If the equations are

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2, \\ a_3x + b_3y &= c_3, \end{aligned} \tag{5-28}$$

and no two of the lines are parallel or coincident, they may be solved for x and y to obtain

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

This point lies on the third line if and only if

$$a_3 \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + b_3 \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = c_3.$$

After clearing of fractions and changing signs, this condition can be written in the form

$$a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_2 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Hence Eqs. (5–28) will have a common solution and the corresponding noncoincident and nonparallel lines will be concurrent when the determinant of the coefficients vanishes. If two of the lines are coincident, this determinant will also vanish.

Problem Set 5–5

1. Sketch the following planes that pass through the origin.

(a) $x + 4y - 2z = 0$ (b) $x + 4y + 2z = 0$
(c) $x - 4y + 2z = 0$ (d) $x - 4y - 2z = 0$

2. By means of the traces in all three coordinate planes sketch the following planes.

(a) $x + 4y + 2z = 6$ (b) $x + 4y - 2z = 6$
(c) $x - 4y + 2z = 6$ (d) $x + 4y - 2z = -6$

3. (a) The prices of three commodities are 4, 3, 5, respectively, and the total expenditure is limited to 100. What is the budget equation? Sketch the corresponding plane.

(b) A decision is made to purchase equal amounts of the second and third commodities and twice this amount of the first commodity. How much of each is purchased? Show the corresponding point in the budget plane as determined by the intersection of three planes.

4. (a) The prices of three commodities are 6, 2, 10, respectively, and the total budget is limited to 120. What is the budget equation? Sketch the corresponding plane.

(b) A decision is made to purchase equal amounts of the first and third commodities and three times this amount of the second commodity. How much of each is purchased? Show the corresponding point in the budget plane as determined by the intersection of three planes.

5. In the following problems, p is the price per unit of a commodity if x units of it are demanded and q is the price per unit of a related commodity. Sketch the corresponding demand planes subject to the restriction that p, x, q are zero or positive. Note that x must decrease as p increases, but that x may increase or decrease as q increases.

(a) $x = 10 - 3p - q$
(b) $x = 10 - 3p + q$, $(q \leqq 20)$
(c) $x = 135 - 9p - 3q$
(d) $x = 135 - 9p + 5q$, $(q \leqq 15)$
(e) $15x + 4p + 2q = 30$
(f) $15x + 4p - 2q = 30$, $(q \leqq 20)$

6. Use the definitions of problem 5 for p, x, q, and let y represent the number of units of the second quantity that are demanded. Solve the following linear equations in x and y for x and y in terms of p and q. In separate diagrams draw the demand planes that give x as a function of p and q, and that give y as a function of p and q.

$$p = 15 - x - y, \qquad q = 5 + x - y$$

7. Proceed as in problem 6 for the following demand laws.

(a) $p = 24 - x - 2y, \quad q = 27 - x - 3y$
(b) $9p = 40 - 5x - y, \quad 9q = 44 - x - 2y$
(c) $p = 41 - 5x - 7y, \quad q = 58 - 7x - 10y$

8. Show that the following sets of equations are inconsistent. Give geometrical interpretations of the systems.

(a)
$$\begin{aligned} 2x + 3y + 4z &= 12 \\ x - 2y + 3z &= 0 \\ 4x + 6y + 8z &= 16 \end{aligned}$$

(b)
$$\begin{aligned} 2x + 3y + 4z &= 12 \\ x - 2y + 3z &= 6 \\ 3x - 6y + 9z &= 10 \end{aligned}$$

(c)
$$\begin{aligned} 2x + 3y + 4z &= 12 \\ 4x + 6y + 8z &= 16 \\ 6x + 9y + 12z &= 20 \end{aligned}$$

(d)
$$\begin{aligned} 2x + 3y + 4z &= 12 \\ x + 2y + 3z &= 0 \\ x - z &= 16 \end{aligned}$$

(e)
$$\begin{aligned} 2x + 3y + 4z &= 12 \\ x + 2y + 3z &= 0 \\ x - y - 3z &= 20 \end{aligned}$$

(f)
$$\begin{aligned} 2x - 3y &= 1 \\ 3x - 2z &= 1 \\ 9y - 4z &= 1 \end{aligned}$$

9. Show that the following sets of equations are dependent. Find the general solution of the systems. Give geometrical interpretations of the systems.

(a) $2x + 3y + 4z = 12$
$x + 2y + 3z = 0$
$4x + 6y + 8z = 24$

(b) $2x + 3y + 4z = 12$
$x + 2y + 3z = 0$
$x - y - 3z = 36$

(c) $2x - 3y = 1$
$3x - 2z = 1$
$-9y + 4z = 1$

(d) $x - 2y + 3z = 6$
$2x + 3y + 4z = 12$
$3x - 20y + 13z = 18$

10. Show that the following systems of equations have no solutions other than (0, 0, 0).

(a) $x - 2y + 3z = 0$
$2x - y + 4z = 0$
$3x + 5y = 0$

(b) $x - 2y + 3z = 0$
$2x + 3y + 4z = 0$
$4x - 2y + 10z = 0$

11. Show that the following systems of equations have solutions other than (0, 0, 0) and find the most general solutions. Illustrate graphically.

(a) $x - 2y + 3z = 0$
$2x - y + 4z = 0$
$2x + 5y = 0$

(b) $x - 2y + 3z = 0$
$2x + 3y + 4z = 0$
$4x - y + 10z = 0$

12. Determine whether or not each of the following systems of equations has a solution. If so, find the solution. If not, find the vertices of the triangle formed by the three lines. Illustrate each problem graphically.

(a) $2x + 4y = 11$
$-5x + 3y = 5$
$x - y = -2$

(b) $2x + 4y = 11$
$-5x + 3y = 5$
$x - y = 2$

(c) $3x - 2y = 1$
$2x + 3y = -8$
$x - 2y = 5$

(d) $3x - 2y = 1$
$2x + 3y = -8$
$x - 2y = 3$

(e) $3x + 4y = 12$
$5x + 12y = 60$
$2x + 2y = 3$

(f) $3x + 4y = 12$
$5x + 12y = 60$
$2x - 2y = 3$

CHAPTER 6

QUADRATIC EQUATIONS

6–1 Square root. If n is a positive number, a *square root* of n is defined as a real number x such that

$$x^2 = n.$$

It is assumed here that there is at least one such number x. Then

$$(-x)^2 = (-1)^2x^2 = n,$$

so that $-x$ is also a square root of n. Further, if y is any number such that $y^2 = n$, then

$$y^2 = x^2, \qquad y^2 - x^2 = 0, \qquad (y - x)(y + x) = 0,$$

so that either $y = x$ or $y = -x$. Hence a given positive number has two and only two square roots. One is positive, is represented by the symbol $\sqrt{n}$, and is called *the (principal) square root.* The other is negative and is represented by the symbol $-\sqrt{n}$. The square root of 0 is 0; zero has only one square root. Note that the square root of a negative number has not been defined, but no real number could be a square root of such a number, since the square of any real number is positive (Section 2–3). In summary,

$$x = \sqrt{n} \quad \text{means} \quad n \geqq 0, \qquad x \geqq 0, \qquad \text{and} \qquad x^2 = n. \tag{6-1}$$

Thus $\sqrt{3}$ is the positive number such that $(\sqrt{3})^2 = 3$; $\sqrt{2 + \sqrt{3}}$ is the positive number such that $(\sqrt{2 + \sqrt{3}})^2 = 2 + \sqrt{3}$; $\sqrt{(-2)^2} = \sqrt{4} = 2$. More generally, $\sqrt{c^2} = c$ if c is positive, and $\sqrt{c^2} = (-c)$, if c is negative. In terms of absolute value,

$$\sqrt{c^2} = |c|. \tag{6-2}$$

The square roots of real numbers are not, in general, rational numbers, that is, quotients of two nonzero integers. $\sqrt{9} = 3$, $\sqrt{9/4} = 3/2$ are rational numbers but $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, . . . are not. Euclid's proof that $\sqrt{2}$ is not rational is essentially as follows. Any integer is either even or odd, that is, of the form $2m$ or $2m + 1$, where m is an integer. If $\sqrt{2} = p/q$, where p and q are integers with no common factor,

$$p^2 = 2q^2.$$

If p is odd, $p = 2p_1 + 1$, then $p^2 = 4p_1^2 + 4p_1 + 1$ is the even number $4(p_1^2 + p_1)$ increased by 1 and hence is odd, while $2q^2$ is even. This is impossible. If p is even, $p = 2p_1$, then q must be odd, else p and q would have 2 as common factor. Hence

$$4p_1^2 = 2q^2 \qquad \text{or} \qquad q^2 = 2p_1^2,$$

where q^2 is odd and $2p_1^2$ is even. This contradiction shows that p cannot be even and completes the proof that $\sqrt{2}$ is not rational.

6–2 Decimal approximation to $\sqrt{n}$. In Section 3–7 it was shown that rational fractions and repeating (or terminating) decimal fractions are equivalent. A positive integer n which is not the square of an integer is such that $\sqrt{n}$ cannot be rational. A generalization of Euclid's method can be used to prove this. In such cases, a satisfactory decimal approximation to $\sqrt{n}$ is sought, noting that the decimal equivalent to $\sqrt{n}$ can neither terminate nor repeat.

Table I, Appendix III, gives, to four significant figures, square roots of integers 1, 2, . . . , 100; 10, 20, 30, . . . , 1000. Square roots of these integers multiplied or divided by 100 are obtained from the table by moving the decimal points one place to the right or left, respectively. For example, the table gives $\sqrt{48} = 6.928$ and $\sqrt{480} = 21.91$. Hence

$$\sqrt{0.48} = \tfrac{1}{10}\sqrt{48} = 0.6928 \qquad \text{and} \qquad \sqrt{4.80} = \tfrac{1}{10}\sqrt{480} = 2.191.$$

For most calculations, four significant figures are sufficient. If great accuracy is required, the table serves as a starting point for the method now to be discussed.

The division process for $\sqrt{n}$. An algorithm for finding the square root of a number begins by writing $x^2 = n$ in the form

$$x = \frac{n}{x}.$$

If the exact value of x were known, then the quotient n/x would also be x. If an approximate value, x_1, of x is known, then

$$q_1 = \frac{n}{x_1}$$

is another approximation. If x_1 is too large, q_1 is too small and vice versa. The arithmetic average of x_1 and q_1,

$$x_2 = \frac{x_1 + q_1}{2},$$

is used as the second approximation.

It can be shown (Appendix II) that if $x_1 > \sqrt{n}$, then $x_2 > \sqrt{n}$ and x_2 is a better approximation to $\sqrt{n}$ than x_1. By repetition of the process, $\sqrt{n}$ can be obtained to any desired number of significant figures. This gives the following rule:

To find $\sqrt{n}$, estimate a first approximation x_1 and divide n by x_1 to obtain q_1. Take the arithmetic average of x_1 and q_1 to obtain x_2. Divide n by x_2 to obtain q_2 and let $x_3 = (x_2 + q_2)/2$. The number of significant figures obtained for q_2 should be twice the number of significant figures to which x_2 and q_2 agree. Continue the operations, obtaining the quotient to twice the number of significant figures to which the divisor and quotient agree. Since the last figure in the final average is doubtful, round off the answer to one less figure.

EXAMPLE 6–1. Find $\sqrt{43}$ and $\sqrt{430}$ to four significant figures.

The details of the divisions are not shown but successive results are these:

$\sqrt{43}$ is between 6 and 7. Take $x_1 = 6.5$.

Then $q_1 = 43/6.5 = 6.6$ and $x_2 = \frac{1}{2}(6.5 + 6.6) = 6.55$.

$43/6.55 = 6.565$, where the quotient is carried to four significant figures because the divisor and quotient agree to two significant figures. Then

$$x_3 = \tfrac{1}{2}(6.550 + 6.565) = 6.557,$$

which agrees with the value in Table I. To find $\sqrt{43}$ more accurately than could befound in the table, divide 43 by 6.557 to find $q_3 = 6.5578771^-$ carried to eight significant figures. Hence $\sqrt{43} = 6.5574385$.

To find $\sqrt{430}$, take $x_1 = 20$. Then $q_1 = 21.5$ and the average is nearer 21 than 20. Use $x_1 = 21$, then $q_2 = 20.48$ and

$$x_3 = \tfrac{1}{2}(21.00 + 20.48) = 20.74,$$

which is slightly too large but correct to four significant figures.

6–3 Algebraic operations with square roots. If a and b are positive numbers and c is any number, then $\sqrt{a}$, $\sqrt{b}$, $|c|$ are positive numbers, and the fundamental laws for combining them are the following:

$$\sqrt{a}\sqrt{b} = \sqrt{ab}, \tag{6–3}$$

$$\sqrt{c^2 a} = |c|\sqrt{a}, \tag{6–4}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b}. \tag{6–5}$$

Proof. Let $x = \sqrt{a}$, $y = \sqrt{b}$, so that by definition $x^2 = a$ and $y^2 = b$. By multiplication,

$$xy = \sqrt{a}\sqrt{b} \qquad \text{and} \qquad x^2y^2 = ab.$$

If the commutative and associative laws are applied to x^2y^2,

$$x^2y^2 = (xy)^2 = ab,$$

so that, by definition, $xy = \sqrt{ab}$. Equation (6-3) then follows from the two forms of xy.

If Eq. (6-3) is applied to $\sqrt{c^2a}$, and Eq. (6-2) is used in the form $\sqrt{c^2} = |c|$, Eq. (6-4) is obtained. It is recognized that if c is positive, then $\sqrt{c^2} = c$, whereas if c is negative, $\sqrt{c^2} = -c$. For example,

$$\sqrt{(-5)^2 2} = \sqrt{50} = 5\sqrt{2}.$$

Equation (6-5) may be derived as follows:

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{a}{b} \cdot \frac{b}{b}} = \sqrt{ab\left(\frac{1}{b}\right)^2} = \left(\frac{1}{b}\right)\sqrt{ab} = \frac{\sqrt{ab}}{b}.$$

Likewise,

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}.$$

These laws indicate that the operations of multiplication and extraction of square roots are commutative. However, the operations of addition and extraction of square roots are not commutative. That is, if a and b are positive numbers, then

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b},$$

where the symbol $\sqrt{a+b}$ means that the addition must be performed first. It is also noted that

$$\sqrt{a^2+b^2} \neq a+b, \qquad \sqrt{a+b} < \sqrt{a} + \sqrt{b},$$

and

$$\sqrt{a^2+b^2} < a+b, \qquad (ab \neq 0).$$

PROBLEM SET 6-1

1. Table I, Appendix III, gives square roots of N and $10N$ for $N = 1$ to 100. On the basis of Eqs. (6-3) and (6-4), justify that the table can be used to find square roots of $N/10$ and $N/100$ by shifting the decimal point.

2. Find the following square roots, using Table I and a convenient shift of decimal point.

(a) $\sqrt{0.39}$ (b) $\sqrt{3.9}$ (c) $\sqrt{0.039}$ (d) $\sqrt{3900}$

3. Find the following square roots to two significant figures by the division process. Check your results by Table I.

(a) $\sqrt{13}$ (b) $\sqrt{1.3}$ (c) $\sqrt{19}$ (d) $\sqrt{1.9}$ (e) $\sqrt{37}$ (f) $\sqrt{3.7}$

4. Use the tables or the division process to find the square roots of the following numbers to four significant figures.

(a) 23 (b) 230 (c) 563 (d) 0.563 (e) 8.479 (f) 23.72

5. Starting with the answers in problem 4, find the square roots to six significant figures.

6. Use the division process to find the square roots of the following numbers to four significant figures.

(a) $\pi = 3.142$ (b) $10\pi = 31.42$ (c) $1/\pi = 0.3183$
(d) $e = 2.718$ (e) $10e = 27.18$ (f) $1/e = 0.3679$

7. Use Table I to find the inner square roots and complete the problem using the division process.

(a) $\sqrt{3+\sqrt{7}}$ (b) $\sqrt{\sqrt{7}+\sqrt{5}}$ (c) $\sqrt{\sqrt{7}-\sqrt{5}}$.

8. Find the square roots of the following fractions, to four significant figures, in the following ways:

(i) rationalize the denominator (Eq. 6–5) first and then extract roots;
(ii) divide first and find the square root of the decimal fraction.

(a) 17/5 (b) 12/7 (c) 37/6 (d) 22/7

9. Prove that $\sqrt{3}$ is irrational. (Suggestion: Every natural number is of the form $3n$, $3n+1$ or $3n+2$.)

10. Prove that $\sqrt{2/3}$, $\sqrt{3/2}$, and $\sqrt{6}$ are irrational.

6–4 Algebraic solution of a quadratic equation. An equation of the form

$$ax^2 + bx + c = 0, \qquad (a \neq 0), \tag{6–6}$$

or one which can be reduced to this form by permissible operations (Section 4–1), is called *a quadratic equation in one unknown*. The coefficients a, b, c are limited in this discussion to real numbers. *A priori* it is not known whether there are real numbers x which satisfy the equation. The finding of such numbers, if they exist, is discussed under three headings: factoring, completing the square, and the quadratic formula.

Factoring. If the coefficients are integers, it may be possible to factor the polynomial $ax^2 + bx + c$ into two linear factors with integral coeffi-

cients (Section 3–9). The equation is then solved by setting each of these linear factors equal to zero (Section 2–2) and solving each such equation for x. Solutions obtained in this way are rational numbers.

EXAMPLE 6–2.

(a) To solve the equation $x^2 - 3x = 0$:

$$x(x - 3) = 0, \qquad x = 0 \qquad \text{and} \qquad x - 3 = 0 \qquad \text{or} \qquad x = 3.$$

(b) To solve the equation $6x^2 + 5x - 4 = 0$:

$$6x^2 + 5x - 4 = (2x - 1)(3x + 4).$$

Hence the solutions are found from the equations

$$2x - 1 = 0 \qquad \text{and} \qquad 3x + 4 = 0 \qquad \text{to be} \qquad x = \tfrac{1}{2}, \qquad x = -\tfrac{4}{3}.$$

(c) To solve the equation $6x^2 + 5x - 2 = 0$:

Examination of all possibilities shows that $6x^2 + 5x - 2$ cannot be factored into linear factors with rational coefficients. This means that the equation $6x^2 + 5x - 2 = 0$ does not have rational solutions, but leaves open the possibility that it does have irrational solutions.

Completing the square. This general method is first illustrated by several examples. When the rational factors cannot be found readily, this procedure is used instead of the factoring method. To solve

$$3x^2 + 4x - 4 = 0,$$

divide by 3 and transpose the constant to the right member

$$x^2 + \tfrac{4}{3}x = \tfrac{4}{3}.$$

Add to each member the square of one-half the coefficient of x, in this case $(2/3)^2$, thus making the left member a perfect square:

$$x^2 + \tfrac{4}{3}x + \tfrac{4}{9} = \tfrac{4}{3} + \tfrac{4}{9} = \tfrac{16}{9}$$

$$(x + \tfrac{2}{3})^2 = \tfrac{16}{9}.$$

Take square roots of both sides:

$$(x + \tfrac{2}{3}) = \pm\tfrac{4}{3}, \qquad x = -\tfrac{2}{3} \pm \tfrac{4}{3}$$

$$x = -2 \qquad \text{and} \qquad x = \tfrac{2}{3}.$$

The original polynomial could have been factored thus:

$$3x^2 + 4x - 4 = (x + 2)(3x - 2) = 0,$$

again yielding the solutions $x = -2$ and $x = 2/3$.

For the equation $6x^2 + 5x - 2 = 0$:

$$x^2 + \frac{5}{6}x = \frac{1}{3}$$

$$x^2 + \frac{5}{6}x + \frac{25}{144} = \frac{1}{3} + \frac{25}{144} = \frac{73}{144}$$

$$\left(x + \frac{5}{12}\right)^2 = \frac{73}{144}$$

$$x + \frac{5}{12} = \frac{\pm\sqrt{73}}{12}$$

$$x = \frac{-5 + \sqrt{73}}{12} \quad \text{and} \quad x = \frac{-5 - \sqrt{73}}{12}.$$

Since these are not rational and $\sqrt{73} = 8.544$, approximately, the approximate values of the roots are 0.2953 and -1.1287.

Quadratic formula. If the method of completing the square, as illustrated above, is applied to the equation $ax^2 + bx + c = 0$, the quadratic formula is obtained.

Divide by a and transpose the constant term:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $b^2/(4a^2)$ to each member:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

If the number $b^2 - 4ac$ is non-negative,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{6–7}$$

This formula can be used to find the real solutions of Eq. (6–6) whenever $b^2 - 4ac \geqq 0$.

EXAMPLE 6–3. For the equation $6x^2 + 5x - 2 = 0$:

$$a = 6,\ b = 5,\ c = -2$$

and Eq. (6–7) gives

$$x = \frac{-5 \pm \sqrt{25 - 4(6)(-2)}}{12} = \frac{-5 \pm \sqrt{73}}{12}.$$

For the equation $3x^2 + 4x - 4 = 0$: $a = 3$, $b = 4$, $c = -4$, and

$$x = \frac{-4 \pm \sqrt{16 + 48}}{6} = \frac{-4 \pm \sqrt{64}}{6} = \frac{-4 \pm 8}{6} = -2 \text{ and } 2/3.$$

6–5 Complex numbers. If the process of completing the square or using the quadratic formula gives $b^2 - 4ac$ negative, the equation does not have real solutions. If n is a positive number, the equation

$$x^2 = -n$$

cannot have real solutions, since the square of any real number is positive. The number system can be extended to include numbers represented by $+\sqrt{-n}$ and $-\sqrt{-n}$. These new numbers are called *imaginary* numbers. If a and b are real numbers, then numbers of the form $a + b\sqrt{-n}$, $n > 0$ are called *complex numbers*. If $b^2 - 4ac < 0$, the solutions of $ax^2 + bx + c = 0$ are complex numbers. In the present text, such solutions are not emphasized.

6–6 Equations involving square roots. The solving of equations which involve the square roots of linear or quadratic polynomials can often be reduced to solving quadratic equations of the type of Eq. (6–6). This is done through a process of squaring to remove the radicals, and hence may yield two equations which are not equivalent. While it is known that if $a = b$, then $a^2 = b^2$, it is also known that when $a^2 = b^2$, then a may equal b or may equal $-b$. Since the squaring process is not a permissible operation, the solutions of an equation resulting from its use may or may not be solutions of the original equation. When this operation is performed, it is an essential part of the problem to find out if solutions obtained are solutions of the original equation. This is done by substitution in the original equation.

Two types of equations are considered: those that involve one square root, and those that involve two square roots. To solve

$$\sqrt{Ax^2 + Bx + C} = Dx + E, \tag{6–8}$$

both members are squared and the resulting quadratic equation is solved

for its roots, r_1 and r_2. Since the left member is always positive, those values of r are retained which make $Dr + E$ positive or zero.

EXAMPLE 6–4. Solve the equation

$$\sqrt{x^2 - 3x + 27} = 2x + 3.$$

Squaring:

$$x^2 - 3x + 27 = 4x^2 + 12x + 9$$

$$3x^2 + 15x - 18 = 0$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0; \quad x = 1 \quad \text{and} \quad x = -6.$$

Since $2(1) + 3 > 0$ and $2(-6) + 3 < 0$, the only solution is $x = 1$.

EXAMPLE 6–5. Solve the equation

$$\sqrt{x^2 + 24x + 3} = 2x + 3.$$

Squaring:

$$x^2 + 24x + 3 = 4x^2 + 12x + 9$$

$$3x^2 - 12x + 6 = 0$$

$$x^2 - 4x + 4 = -2 + 4$$

$$(x - 2)^2 = 2$$

$$x = 2 \pm \sqrt{2}.$$

Since $2(2 \pm \sqrt{2}) + 3$ are both positive, $x_1 = 2 + \sqrt{2}$ and $x_2 = 2 - \sqrt{2}$ are both solutions of the equation. A further check could be obtained as follows:

If $x_1 = 2 + \sqrt{2}$, then

$$(2 + \sqrt{2})^2 + 24(2 + \sqrt{2}) + 3 = 57 + 28\sqrt{2}$$

and

$$[2(2 + \sqrt{2}) + 3]^2 = [7 + 2\sqrt{2}]^2 = 57 + 28\sqrt{2}.$$

A similar calculation shows that $x_2 = 2 - \sqrt{2}$ is also a solution.

To solve an equation of the form

$$\sqrt{ax + b} + \sqrt{cx + d} = e, \tag{6–9}$$

isolate one radical and square both sides to get an equation with one radical. Isolate this radical and square to obtain a quadratic equation.

The roots of this equation may or may not be solutions of the original equation, and each solution must be tested by substitution in the original equation. This substitution is an *essential* part of the solution process.

EXAMPLE 6–6. Find the real solutions of the equation

$$\sqrt{3x+1} + \sqrt{x-4} = 3.$$

Solution: $\sqrt{3x+1} = 3 - \sqrt{x-4}$

$$3x + 1 = 9 - 6\sqrt{x-4} + x - 4$$

$$6\sqrt{x-4} = 4 - 2x.$$

At this point it may be observed that x must be $\geqq 4$ if the left member is to be real, and that $x \leqq 2$ if the right member is to be positive.

$$36(x-4) = 16 - 16x + 4x^2.$$

Divide by 4 and collect terms:

$$x^2 - 13x + 40 = 0$$

$$(x-5)(x-8) = 0$$

$$x = 5 \quad \text{and} \quad x = 8.$$

These are the solutions of this last quadratic equation, but they may not be solutions of the original equation. Substitution gives

$$\sqrt{16} + \sqrt{1} \neq 3 \quad \text{and} \quad \sqrt{25} + \sqrt{4} \neq 3.$$

Hence, the original equation has no solutions.

The equation $\sqrt{3x+1} - \sqrt{x-4} = 3$ is a modification of the foregoing and has $x = 5$, $x = 8$ as solutions.

Equations involving absolute values of linear functions are similar to those involving square roots, in view of the special form of Eq. (6–4)

$$|c| = \sqrt{c^2}.$$

To solve such an equation, replace each expression of the form $|ax + b|$ by $\sqrt{(ax+b)^2}$, and proceed as before. Since the squaring process may introduce extraneous solutions, substitution in the original equation is an essential part of the solution process.

EXAMPLE 6–7. Find solutions, if any, for the equation

$$|x| - |x - 8| = 6.$$

The equation can be written in the form

$$\sqrt{x^2} - 6 = \sqrt{(x-8)^2}.$$

Squaring:

$$x^2 - 12\sqrt{x^2} + 36 = x^2 - 16x + 64$$

Collecting terms:

$$12\sqrt{x^2} = 16x - 28$$

$$3\sqrt{x^2} = 4x - 7$$

Squaring:

$$9x^2 = 16x^2 - 56x + 49$$

$$x^2 - 8x + 7 = 0$$

$$(x-1)(x-7) = 0$$

so that the solutions of the final quadratic equation are $x = 1$ and $x = 7$.

If $x = 1$, $\quad |1| - |1-8| = 1 - 7 = -6 \neq 6.$

If $x = 7$ $\quad |7| - |7-8| = 7 - 1 = 6.$

Hence the only solution of the original equation is $x = 7$.

EXAMPLE 6–8. If $A(0)$ and $B(6)$ are points on the number line, determine the points $X(x)$ such that $AX = \frac{1}{2}XB$. (See Sections 1–7 and 4–3.)

The equation to be solved is

$$|x| = \tfrac{1}{2}|x-6| \quad \text{or} \quad \sqrt{x^2} = \tfrac{1}{2}\sqrt{(x-6)^2}$$

$$x^2 = \tfrac{1}{4}(x^2 - 12x + 36)$$

$$3x^2 + 12x - 36 = 0$$

$$x^2 + 4x - 12 = 0$$

$$(x-2)(x+6) = 0$$

$$x = 2 \quad \text{and} \quad x = -6.$$

If $x = 2$, $\quad |2| = 2 \quad$ and $\quad \tfrac{1}{2}|2-6| = \tfrac{1}{2}\cdot 4 = 2;$

If $x = -6$, $\quad |-6| = 6 \quad$ and $\quad \tfrac{1}{2}|-6-6| = \tfrac{1}{2}\cdot 12 = 6.$

Hence $x = 2$ and $x = -6$ are both solutions of the problem.

PROBLEM SET 6–2

1. Solve each of the following equations by factoring, if possible; if rational factors are not found, solve the problem by *completing the square.*

(a) $2x^2 + 5x = 0$
(b) $5x^2 - 2x = 0$
(c) $4x^2 + 4x + 1 = 0$
(d) $x^2 - 6x + 9 = 0$
(e) $5x^2 - 4x - 28 = 0$
(f) $2x^2 - x - 1 = 0$
(g) $x^2 + 2x - 6 = 0$
(h) $x^2 + 8x + 2 = 0$
(i) $3x^2 + 8x + 1 = 0$
(j) $5x^2 + 12x - 4 = 0$

2. Use the quadratic formula to determine whether or not the following equations have real roots. If the roots are real, find decimal approximations to the roots.

(a) $x^2 + 4x + 7 = 0$
(b) $x^2 + 4x - 7 = 0$
(c) $2x^2 + 3x - 4 = 0$
(d) $2x^2 + 3x + 4 = 0$
(e) $3x^2 - 5x - 1 = 0$
(f) $3x^2 - 5x + 1 = 0$
(g) $3x^2 - 5x + 3 = 0$
(h) $3x^2 - 5x - 3 = 0$

3. A variation in the method for solving a quadratic equation of the form $ax^2 + bx + c = 0$ begins by multiplying the equation by a and then completing the square. Use this method to solve the following equations.

(a) $2x^2 - x - 1 = 0$
(b) $3x^2 + 8x + 1 = 0$
(c) $2x^2 + 3x - 4 = 0$
(d) $2x^2 + 3x + 4 = 0$
(e) $3x^2 - 5x + 3 = 0$
(f) $3x^2 - 5x - 3 = 0$
(g) $ax^2 + bx + c = 0$

4. Find the solutions, if any, of the following equations.

(a) $\sqrt{2x + 4} = -2$
(b) $\sqrt{2x + 4} = 2$
(c) $\sqrt{2x + 4} = 3$
(d) $\sqrt{8x + 9} = x + 2$
(e) $\sqrt{8x + 25} = x + 2$
(f) $\sqrt{8x + 25} = 2 - x$
(g) $\sqrt{ax + b} = \sqrt{cx + d}$, $(a \neq c)$. Justify the answer.

5. Find the solutions, if any, of the following equations.

(a) $\sqrt{x^2 - 5x + 31} = 2x + 1$
(b) $\sqrt{x^2 - 5x + 31} = -1 - 2x$
(c) $\sqrt{x^2 + 24x + 3} = 2x + 3$
(d) $\sqrt{x^2 + 30x + 84} = 2x + 9$
(e) $\sqrt{x^2 - 6x} = 10 + 3x$
(f) $\sqrt{x^2 - 4x - 3} = 1 + 2x$

6. Find the solutions, if any, of the following equations.

(a) $\sqrt{3x + 1} + \sqrt{x - 4} = 5$
(b) $\sqrt{3x + 1} - 2\sqrt{x - 4} + 1 = 0$
(c) $\sqrt{3y + 7} - \sqrt{2y + 3} = 1$

(d) $\sqrt{3x+4} - \sqrt{2x-4} = 2$

(e) $\sqrt{3x+4} - \sqrt{2x-4} = 6$

(f) $\sqrt{4x+2} - \sqrt{4x-2} = 2$

7. Find the solutions, if any, of the following equations.

(a) $|x-4| = 3$ (b) $|x-4| = -3$

(c) $|x+2| = 3$ (d) $|4x+1| = 11$

(e) $|ax-b| = c,\ a \neq 0$

8. Find the solutions, if any, of the following equations.

(a) $|x-2| = |x-4|$ (b) $|x-2| = \frac{1}{2}|x-6|$

(c) $|2x-3| = |4+x|$ (d) $|x-2| = 2|x-6|$

(e) $|x-a| = |x-b|,\ a \neq b$. What can be said about the solution if $a = b$?

9. If $A(a)$ and $B(b)$ are two distinct points on the number line, (a) prove that there is one and only one point X on the line such that $AX = BX$; (b) prove that there are two and only two points X on the line such that $AX = k \cdot BX$, where k is a positive number different from 1.

10. Find the solutions, if any, of the following equations:

(a) $|x+2| + |x-2| = 6$ (b) $|x+2| + |x-2| = 4$

(c) $|x+2| + |x-2| = 3$ (d) $|x+2| - |x-2| = 3$

(e) $|x| + |x-8| = 12$ (f) $|x| + |x-8| = 8$

(g) $|x-8| + |x| = 6$ (h) $|x-8| - |x| = 6$

6–7 The quadratic function and its graph. If y is defined by the equation

$$y = ax^2 + bx + c, \qquad (a \neq 0), \tag{6–10}$$

y is a *quadratic function* of x. For each value of x there is one and only one value of y (see Section 4–6). The symbols a, b, c are real constants, and the domain of the variable x is the set of all real numbers unless specific limitations are stated or implied by the nature of the problem. The set of all points which correspond to number pairs (x, y) that satisfy Eq. (6–10) is called the graph of this equation. If it is not convenient to select the same size units on both axes, it is understood that distances are measured only along lines parallel to the axes. In this case, the diagram is drawn in what is called the affine plane, distinguishing it from the ordinary or Euclidean plane in which there is a universal unit of measure.

The graph of Eq. (6–10) is known as a *parabola*. It may be sketched by plotting points obtained from assigned values of x and the corresponding computed values of y, then connecting the points in order of increasing x. For the purpose of sketching the curve, certain points are more important than others. These include the *intercepts* on the axes, and the

vertex. To find the intercepts, set $x = 0$, to find $y = c$; set $y = 0$ and solve the equation $ax^2 + bx + c = 0$. If the roots are real, the corresponding points are located. If the roots are not real, no points on the graph are obtained. The *vertex* is the highest or lowest point of the parabola and may be found by completing the square on the right side of Eq. (6–10). The vertical line through the vertex is the axis of symmetry of the parabola, and this is useful in drawing the curve, especially when the curve does not cross the x-axis. The procedures are illustrated by examples.

EXAMPLE 6–9. Sketch the parabola $y = 8 + 2x - x^2$.

By factoring,

$$y = (2 + x)(4 - x).$$

By completing the square,

$$y = 8 - (x^2 - 2x \quad) = 8 + 1 - (x^2 - 2x + 1)$$
$$= 9 - (x - 1)^2.$$

The last form shows two things: first, the largest value that y can have is $y = 9$, and this value occurs when $x = 1$; second, the vertical line $x = 1$ is a line of symmetry, since the two values $x = 1 + c$ and $x = 1 - c$ give the same value of y. If the original quadratic polynomial in x is not readily factored, this last form can be used to find the x-intercepts. The first four points in the following table are sufficient to make a sketch. Several others are added for illustration.

x	0	1	4	−2	5	−3	2
y	8	9	0	0	−7	−7	8

The points $(2, 8)$ and $(-3, -7)$ are obtained from the symmetry about the line $x - 1$ (Fig. 6–1).

EXAMPLE 6–10. Sketch the parabola $y = 4x^2 + 12x + 17$.
By completing the square,

$$y = 4(x^2 + 3x \quad) + 17 = 4(x^2 + 3x + \tfrac{9}{4}) + 17 - 9$$
$$= 4(x + \tfrac{3}{2})^2 + 8.$$

This form shows that the smallest value that y can have is 8, and this value occurs for $x = -3/2$. It also shows that the curve does not cross the x-axis. If $x = 0$, $y = 17$ and the point $(-3, 17)$ is also obtained by using the fact $x = -3/2$ is the axis of symmetry. The sketch is completed using the points $(1, 33)$ and $(2, 57)$ after adopting units to fit the data (Fig. 6–2).

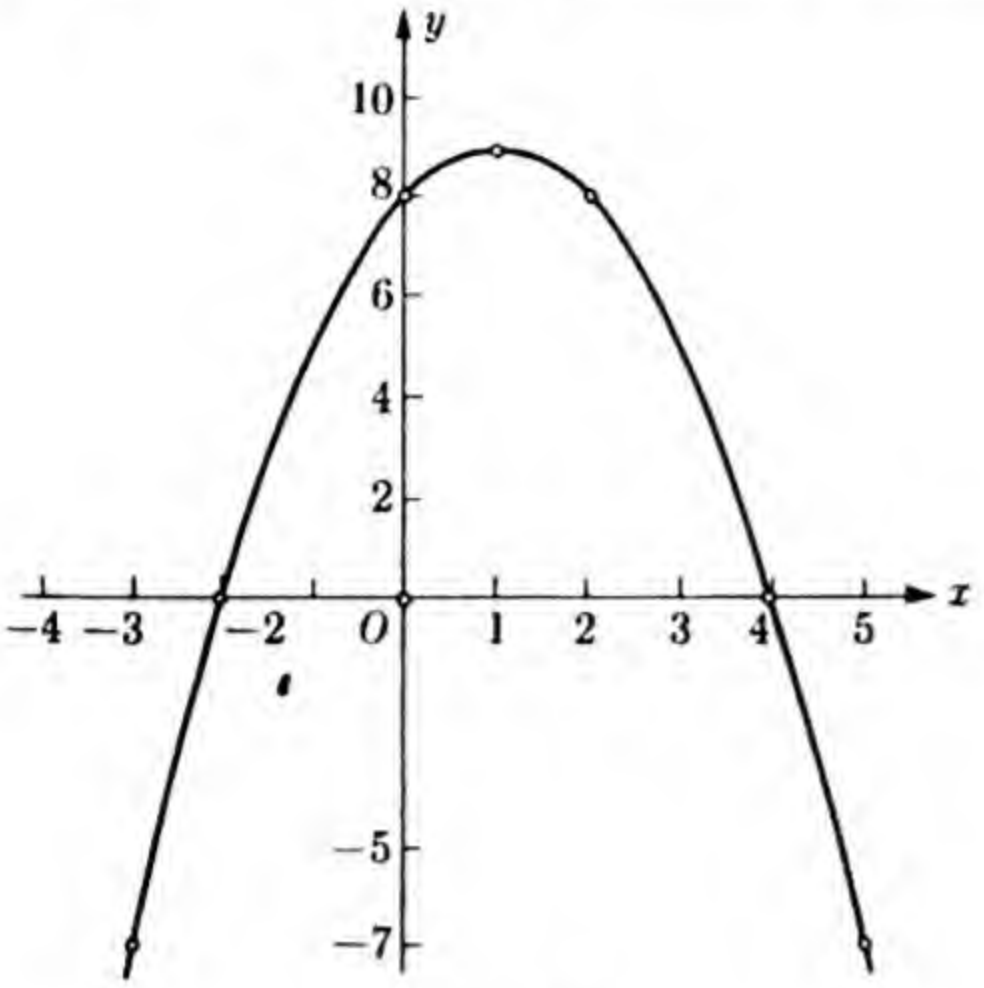


FIGURE 6–1

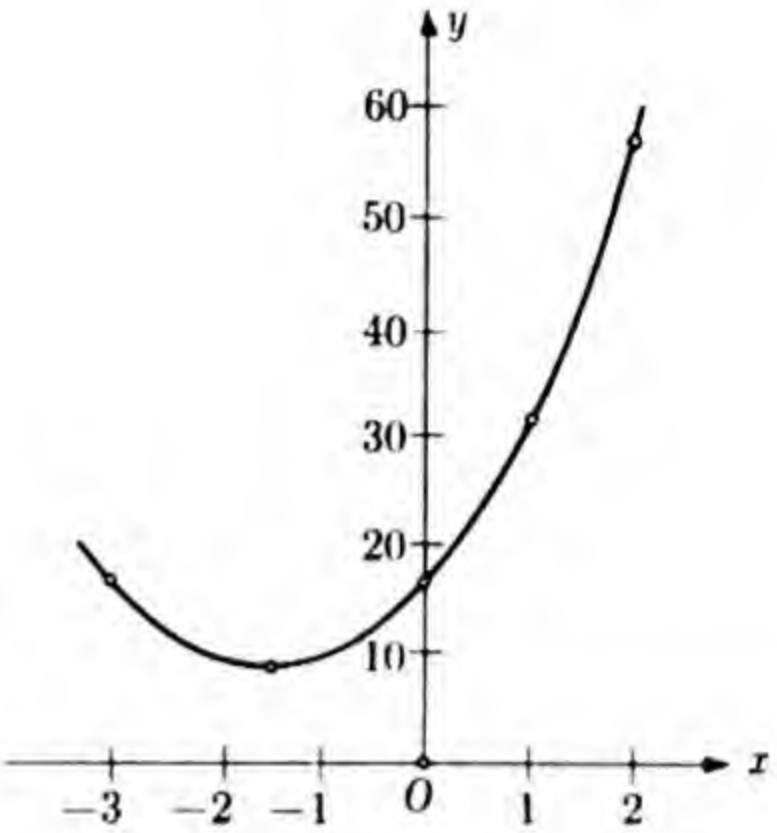


FIGURE 6–2

The method of completing the square applies to the general parabola $y = ax^2 + bx + c$ to give

$$y = a\left(x^2 + \frac{b}{a}x \quad\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

If a is positive, the smallest value that y can have occurs when $x = -b/(2a)$, and $x = -b/(2a)$ is the axis of symmetry. The curve crosses the x-axis if $b^2 - 4ac$ is positive and does not cross this axis if $b^2 - 4ac$ is negative. It is tangent to the x-axis if $b^2 - 4ac = 0$.

If a is negative, the largest value that y can have occurs when $x = -b/(2a)$. The curve crosses the x-axis if $b^2 - 4ac$ is positive. The curve does not cross the x-axis if $b^2 - 4ac$ is negative, and touches the x-axis if $b^2 - 4ac = 0$.

The graph of the equation

$$x = Ay^2 + By + C, \qquad (A \neq 0), \tag{6–11}$$

in which x is a quadratic function of the independent variable y, is a parabola whose axis is the horizontal line $y = -B/(2A)$. The curve is best sketched from its vertex and its intercepts $(C, 0)$ and the values of y obtained by solving the equation $Ay^2 + By + C = 0$. The vertex corresponds to the smallest or largest value that x can have, according as A is positive or negative.

A special case of this equation is met in the form

$$ky = \sqrt{C - x}, \qquad (C > 0, k > 0), \tag{6-12}$$

which leads to the equations

$$k^2y^2 = C - x \qquad \text{or} \qquad x = Ay^2 + C, (y \geqq 0).$$

Here $A = -k^2$ is negative, so that the graph has a maximum value of x. This fact, together with the point $x = 0$, $y = \sqrt{C}/k$, is usually sufficient for a sketch. If additional points are needed, they are computed from Eq. (6–12).

EXAMPLE 6–11. Sketch the part of the parabola corresponding to

$$2y = \sqrt{4 - x}, \; x \geqq 0.$$

An equivalent form is

$$x = 4 - 4y^2, \qquad (x \geqq 0, y \geqq 0).$$

If $x = 0$, $y = 1$; if $y = 0$, $x = 4$, and this is the largest value that x can have. Additional points on the curve are computed to be $x = 1$, $y = \sqrt{3}/2$ and $x = 3$, $y = \frac{1}{2}$. Symmetry can be used to draw the part of the parabola below the x-axis, and negative values of x can be used to find further parts of the parabola. Under the given restrictions, the graph of the equation is confined to the first quadrant.

Given three noncollinear points, there exists in general a unique parabola of the form (6–10), and a unique parabola of the form (6–11), which passes through these points. If the points are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ and the parabola has the form $y = ax^2 + bx + c$, the corresponding

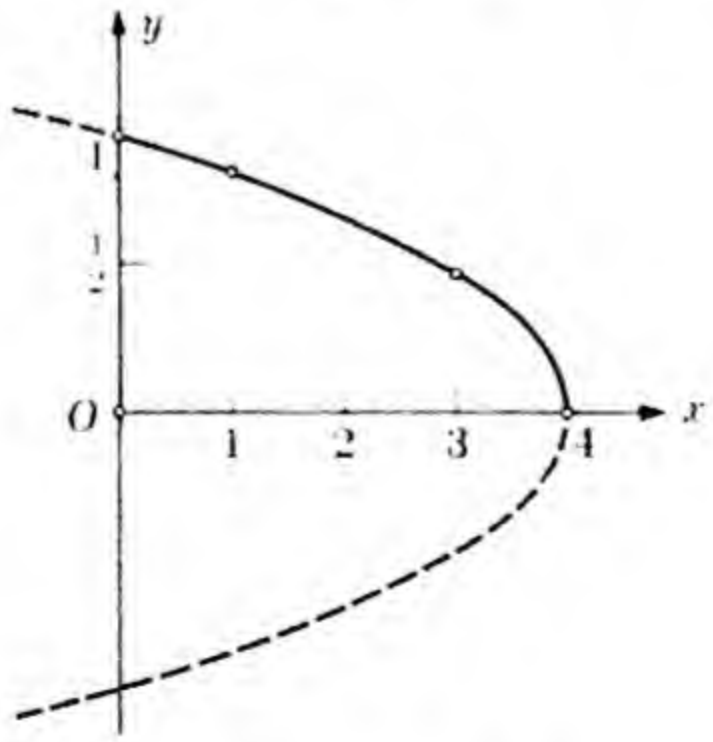


FIGURE 6–3

conditions to determine a, b, c are

$$\begin{aligned} ax_1^2 + bx_1 + c &= y_1, \\ ax_2^2 + bx_2 + c &= y_2, \\ ax_3^2 + bx_3 + c &= y_3. \end{aligned} \tag{6-13}$$

These three linear equations in a, b, c can be solved by the method of successive elimination, or by using third-order determinants and Cramer's rule (Section 5–8). There is a unique solution provided the determinant of the coefficients of a, b, c does not vanish. This determinant is

$$\begin{aligned} D = \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} &= \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 - x_1^2 & x_2 - x_1 & 0 \\ x_3^2 - x_1^2 & x_3 - x_1 & 0 \end{vmatrix} \\ &= (x_2^2 - x_1^2)(x_3 - x_1) - (x_3^2 - x_1^2)(x_2 - x_1) \\ &= (x_2 - x_1)(x_3 - x_1)[(x_2 + x_1) - (x_3 + x_1)] \\ &= (x_2 - x_1)(x_3 - x_1)(x_2 - x_3). \end{aligned} \tag{6-14}$$

This determinant is different from zero if the x's are all different. This is consistent with the fact that the line $x = k$ can cut the parabola in only one point, so that there is no parabola of this form through two different points with the same x. A discussion for the parabola of the form $x = Ay^2 + By + C$ through the points is similar to that given above.

EXAMPLE 6–12. Find the equation of the parabola of the form $y = ax^2 + bx + c$ which passes through the three points $(-2, 0)$ $(2, 8)$, $(5, -7)$. Find the intercepts and vertex of this parabola and sketch the curve.

The equations to be solved are

$$\begin{aligned} 4a - 2b + c &= 0, \\ 4a + 2b + c &= 8, \\ 25a + 5b + c &= -7. \end{aligned}$$

They are solved by first eliminating c:

$$\begin{aligned} 4b &= 8, \\ 21a + 7b &= -7. \end{aligned}$$

Hence $b = 2; 21a + 14 = -7, 21a = -21, a = -1; -4 - 4 + c = 0$, $c = 8$. The equation of the parabola is $y = -x^2 + 2x + 8$. The graph of this curve was discussed in Example 6–9 and shown in Fig. 6–1.

EXAMPLE 6–13. Show that there is no parabola of the form $x = Ay^2 + By + C$ which passes through the points $(-2, 0)$, $(4, 0)$, $(2, 8)$.

The equations to be solved are

$$C = -2,$$
$$C = 4,$$
$$64A + 8B + C = 2.$$

These equations are inconsistent.

PROBLEM SET 6–3

1. Sketch each of the following parabolas from a consideration of its intercepts, vertex, axis of symmetry. Use a few other points if needed.

(a) $y = x^2 - 6x$ (b) $y = 4x - x^2$
(c) $y = x^2 + 4x$ (d) $y = 9 - 4x^2$
(e) $y = 9 + 4x^2$ (f) $y = 3x^2 - 6$
(g) $y = (x - 4)^2$ (h) $y = \frac{1}{4}(x^2 + 4x + 4)$

2. Sketch each of the following parabolas from a consideration of its intercepts, vertex, axis of symmetry. Use a few other points if needed.

(a) $y = x^2 - 4x - 5$ (b) $y = -x^2 + 5x - 4$
(c) $y = 9 + 3x - 2x^2$ (d) $y = 27 - 12x - 4x^2$
(e) $y = x^2 - 5x + 7$ (f) $y = \frac{1}{4}(x^2 + 4x + 6)$
(g) $y = x^2 + 3x - 6$ (h) $y = -x^2 + 6x - 6$

3. Sketch each of the following parabolas from a consideration of its intercepts, vertex, axis of symmetry. Use a few other points if needed.

(a) $x = y^2 - 4y$ (b) $x = y^2 + 4y$
(c) $x = 12y - 6y^2$ (d) $x = (y - 2)^2$
(e) $x = y^2 - 4y + 6$ (f) $x = y^2 - 4y + 2$

4. Sketch each of the following parabolas from a consideration of its intercepts, vertex, axis of symmetry. Indicate the part of the parabola which corresponds to the given equation.

(a) $y = \sqrt{4x + 8}$ (b) $y = \sqrt{2x + 6}, \quad (x \geqq 0)$
(c) $y = \sqrt{9 - 2x}, \quad (x \geqq 0)$ (d) $y = -1 + \sqrt{9 - 4x}, \quad (x \geqq 0)$

5. Find the equations of the parabolas of the form $y = ax^2 + bx + c$ and $x = Ay^2 + By + C$ which pass through the three points $(0, 4)$, $(1, 1)$, $(2, 0)$. Sketch both parabolas in the same diagram using the given points, the intercepts, and the vertices.

6. Proceed as in problem 5 for the three points.

(a) (1, 2) (2, 3) (3, 6) (b) (0, 9), (1, 4), (3, 0)
(c) (0, 3), (1, 1), (3, 0) (d) (0, 1), (2, 2), (4, 5)

7. Find the equation of the parabola of the forms $y = \sqrt{a + bx}$ which is determined by the points (4, 5), (10, 7). Sketch the parabola using the given points and the intercepts on both axes. Where is the vertex? What part of the parabola corresponds to the given equation?

8. Find the equation of the parabola of the form $x = Ay^2 + By + C$ which passes through the points (0, 2), (7, 3), and (27, 5). Compare the graph with that of the parabola of the form $y = \sqrt{a + bx}$ which passes through the points (7, 3) and (27, 5).

9. Find the equation of the parabola of the form $y = \sqrt{c - dx}$ which is determined by the points (2, 4), (−4, 8). Sketch the parabola using the given points, the intercepts on both axes, and the vertex.

10. (a) Show that the three points (1, 5), (2, 7), (−2, −1) lie on a line. (b) Show by means of a third-order determinant that an attempt to find a parabola of the form $y = ax^2 + bx + c$ through these three points leads to $a = 0$, and hence to a line rather than a parabola. (c) If (x_1, y_1), (x_2, y_2), (x_3, y_3) are three points on a line with distinct values of x, prove by means of a third-order determinant that an attempt to find a parabola of the form $y = ax^2 + bx + c$ through these three points leads to $a = 0$.

6-8 Applications. The economic laws of demand, of supply, of total cost, of revenue, etc., can often be approximated by quadratic functions and represented graphically by arcs of parabolas. In such applications the variables are usually zero or positive and still other restrictions are imposed by the economic situation.

Laws of demand and supply. If x represents the quantity demanded or supplied and p represents the price per unit quantity, then for a *demand* law p must be a monotomically decreasing function of x within the first quadrant. The price p may be given by an equation of the form

$$p = ax^2 + bx + c. \tag{6-15}$$

If *a is negative*, then the vertex corresponds to the largest value that p can have and b must also be negative. The vertex corresponds to a negative value of x and although the part of the parabola corresponding to negative values of x has no economic significance, the vertex is a useful guide in drawing the curve. (The simplest case corresponds to $b = 0$.) The variable x is limited to the domain $0 \leqq x \leqq x_1$, where x_1 is the positive root of the equation $ax^2 + bx + c = 0$. If *a is positive* and x_1 and x_2 are the positive roots of $ax^2 + bx + c = 0$, then the domain of x is $0 \leqq x \leqq x_1$, where $x_1 \leqq x_2$. The simplest case is when $x_1 = x_2$:

$$p = a(x - r)^2, \qquad (0 \leqq x \leqq r). \tag{6-16}$$

The quantity x may be given by an equation of the form

$$x = Ap^2 + Bp + C.$$

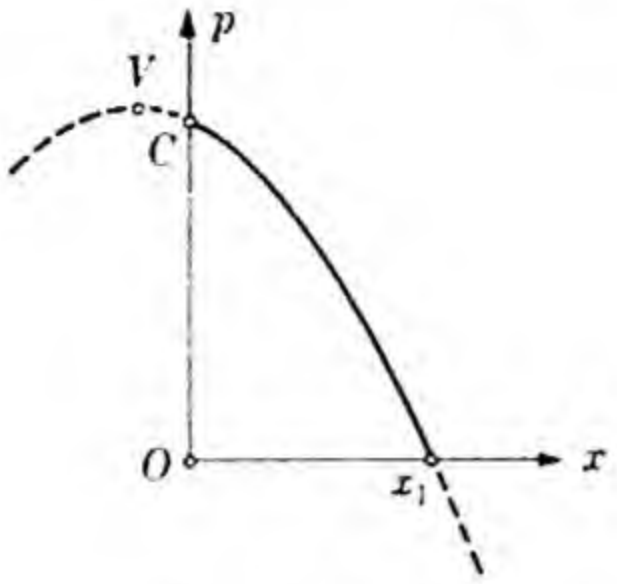


FIGURE 6–4

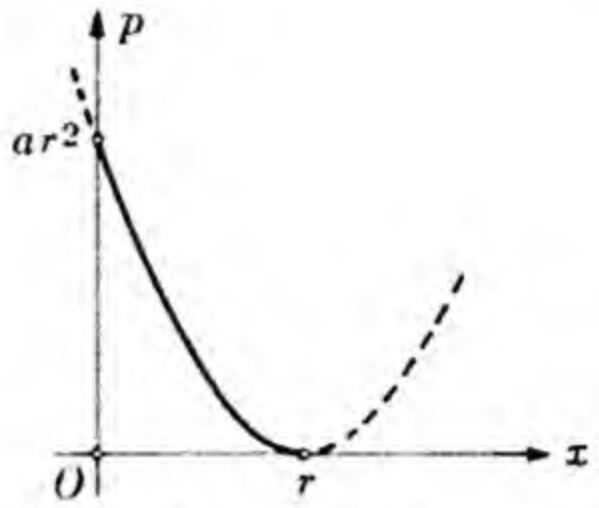


FIGURE 6–5

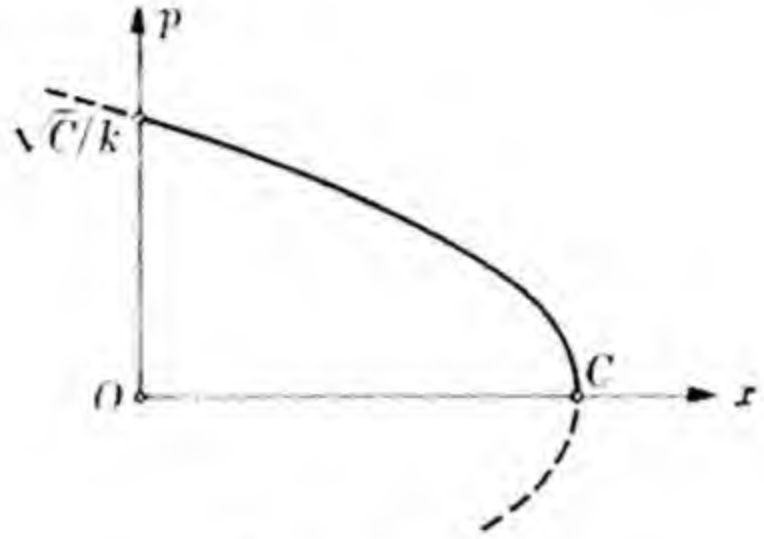


FIGURE 6–6

An important special case corresponds to

$$kp = \sqrt{C - x}, \qquad (0 \leqq x \leqq C), \tag{6–17}$$

or the equivalent equation

$$x = C - k^2p^2, \qquad \left(0 \leqq p \leqq \frac{\sqrt{C}}{k}\right).$$

Graphs corresponding to Eqs. (6–15), (6–16), (6–17), respectively, are shown in Figs. 6–4, 6–5, 6–6.

A *supply* law can be represented in a fashion analogous to that used for demand laws. The price p is usually a monotonically increasing function of x and the upper bound on x depends upon the particular problem. Parabolas of the form

$$x = Ap^2 + Bp + C, \qquad (A \geqq 0), \tag{6–18}$$

where B and C are selected so that the equation $Ap^2 + Bp + C = 0$ has at least one positive or zero root, are often used. An equivalent form is

$$p = d + \sqrt{mx + n}, \qquad (m, n \text{ positive}). \tag{6–19}$$

EXAMPLE 6–14. Sketch the supply curve corresponding to the equation $x = \frac{1}{9}(p^2 + 2p - 3)$, $(0 \leqq x \leqq 5)$.

The given equation can be written in several convenient forms:

$$\begin{aligned} 9x &= p^2 + 2p - 3 \\ &= (p + 3)(p - 1) \\ &= (p^2 + 2p + 1) - 4 \\ &= (p + 1)^2 - 4 \\ (p + 1)^2 &= 9x + 4 \\ p &= -1 + \sqrt{9x + 4}, \end{aligned}$$

where the + sign is selected with the square root in order that p increase with x. From the various forms, the intercepts (0, 1), (0, −3), (−1/3, 0) and the vertex (−4/9, −1) are found. Additional points on the curve may be found by assigning values to p and computing x or assigning value to x and computing p:

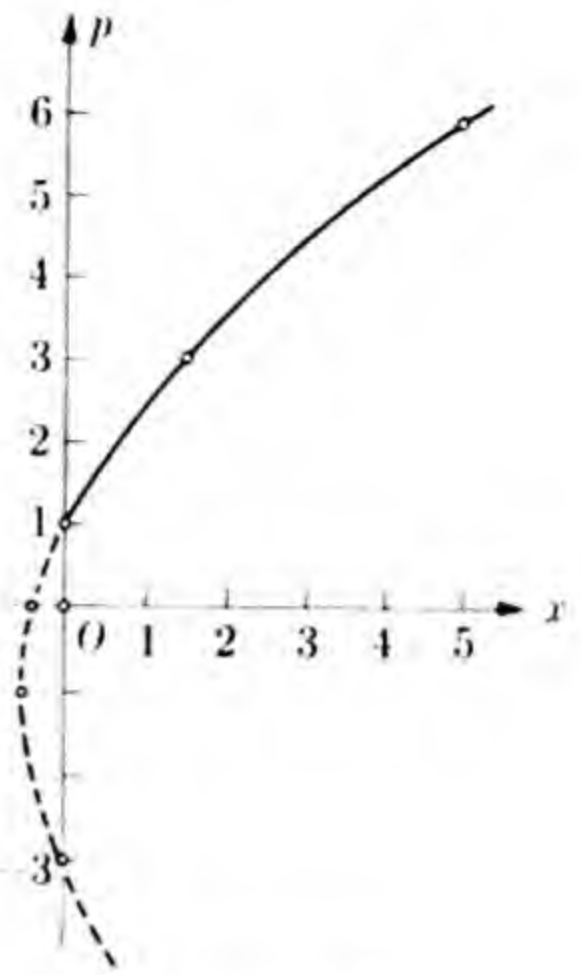


FIGURE 6-7

$x = 5$ gives $p = 6$ and $p = 3$ gives $x = 4/3$ (Fig. 6–7).

Total cost. The total cost Q of producing (and marketing) x units of a commodity can often be approximated by parabolic laws which are similar to, and related to, supply laws. The total cost may involve an initial overhead, and usually the cost of producing one additional unit of the commodity decreases as x increases. Parabolas of the type suggested for supply curves are useful for total cost curves.

Revenue. The total revenue obtained from selling x units of a commodity at price p per unit is $R = px$. If the demand law is linear, so that $p = p_0 - mx$, $(p_0 > 0; m \geqq 0)$, then

$$R = x(p_0 - mx). \tag{6–20}$$

The graph is a simple parabola with intercepts at the origin and at $x = p_0/m$, $p = 0$. A good sketch can be obtained from these two points and the vertex of the parabola. It is not difficult, using the method of completing the square, to show that the vertex corresponds to the value $x = \frac{1}{2}(p_0/m)$.

If the variable p is taken as the independent variable, and the linear demand law is written as $x = x_0 - kp$, then

$$R = p(x_0 - kp) \tag{6–21}$$

and the graph in the pR-plane is a parabola. If the p-axis is now taken horizontal and the R-axis vertical, there is essentially no distinction between the curves drawn for Eqs. (6–20) and (6–21). If, however, the p-axis is retained as vertical and the R-axis is horizontal, then the axis of the parabola (6–21) is also horizontal.

Problem Set 6–4

1. If the demand law is given by the equation $p = b - mx$, $(b \geqq 0, m \geqq 0)$, use the method of completing the square to show that the revenue R has its maximum value for $x = \frac{1}{2}(b/m)$, and determine the value of this maximum revenue.

2. Consider the revenue R as a function of x and sketch the revenue curves corresponding to the following linear demand functions making use of the intercepts and vertex. Use a few other points if necessary.

(a) $p = 36 - 4x$ (b) $3p + 2x = 27$

(c) $3p = 105 - x$ (d) $5x + 3p = 12$

3. Consider the revenue R as a function of p and sketch the revenue curves corresponding to the following linear demand functions. Use intercepts, vertex, and a few other points, if necessary.

(a) $x = 100 - 5p$ (b) $5x + 3p = 30$

4. Sketch the following demand curves. Obtain enough information about the whole parabola to correctly represent the part in the first quadrant.

(a) $p = 48 - 3x^2$ (b) $p = 20 - 4x^2$

(c) $p = 45 - 6x - 3x^2$ (d) $p = 16 - 8x - 4x^2$

(e) $p = x^2 - 10x + 25$ (f) $p = x^2 - 8x + 15$

(g) $x = 36 - p^2$ (h) $x = 32 - 4p - p^2$

(i) $p = \sqrt{9 - 2x}$ (j) $p = \sqrt{16 - 3x}$

5. Sketch the following supply curves and total cost curves. Obtain enough information about the whole parabola so that the part in the first quadrant is represented correctly.

(a) $4x = p^2 - 16, \quad (x \leqq 8)$ (b) $4x = p^2 - 4p, \quad (x \leqq 8)$

(c) $p = \sqrt{9 + 2x}, \quad (x \leqq 8)$ (d) $2x = p^2 + 2p - 8, \quad (x \leqq 8)$

(e) $Q = \sqrt{x + 8}, \quad (x \leqq 8)$ (f) $Q = \sqrt{4x + 9}, \quad (x \leqq 10)$

(g) $Q = 5 + \frac{1}{2}x - \frac{x^2}{50}, \quad (x \leqq 10)$

(h) $Q = \frac{3}{4}x + \frac{1}{16}x^2, \quad (0 \leqq x \leqq 4)$ and

$Q = \sqrt{10x - 24}, \quad (4 \leqq x \leqq 12)$

6–9 Circle and ellipse. In the plane of common experience, called the Euclidean plane, there is a universal unit to measure distances along any line. The distance between two points is computed by use of the Theorem of Pythagoras. A *circle* is defined as the locus of a point which moves so that its distance from a fixed point, the center, is a positive constant, called the radius.

If the center of the circle is the origin and $P(x, y)$ is a variable point on the circle such that $OP = r$, then (Fig. 6–8) from the right triangle OMP,

$$x^2 + y^2 = r^2, \tag{6-21}$$

and conversely, if $P(x, y)$ satisfies this equation, then $OP = r$.

The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6-22}$$

where a and b are positive constants, is called an *ellipse*. It is related to the circle $x^2 + y^2 = a^2$ in such a way that if $P(x, y)$ is a point on this circle and $Q(x, y_1)$, with the same x, is a point on the ellipse, then $y_1 = (b/a)y$. The value of b may be less than a or greater than a, and both possibilities are shown in Fig. 6–8. The graph of Eq. (6–22) is readily sketched from the intercepts on both coordinate axes $(\pm a, 0)$, $(0, \pm b)$. If more precision is needed, the related circle can be used.

If the centers of the circle and ellipse are at the point $C(h, k)$, then the Theorem of Pythagoras applied to the triangle CMP (Fig. 6–9) shows that the equations of the circle and ellipse are:

$$(x - h)^2 + (y - k)^2 = r^2 \tag{6-23}$$

and

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1. \tag{6-24}$$

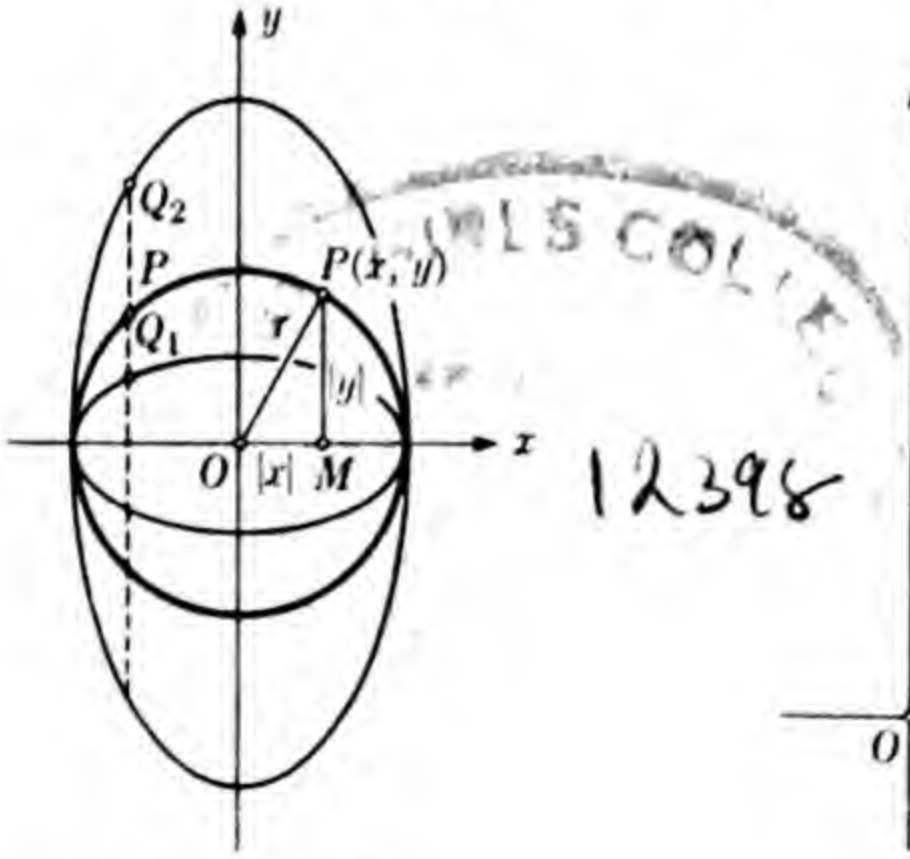


FIGURE 6–8

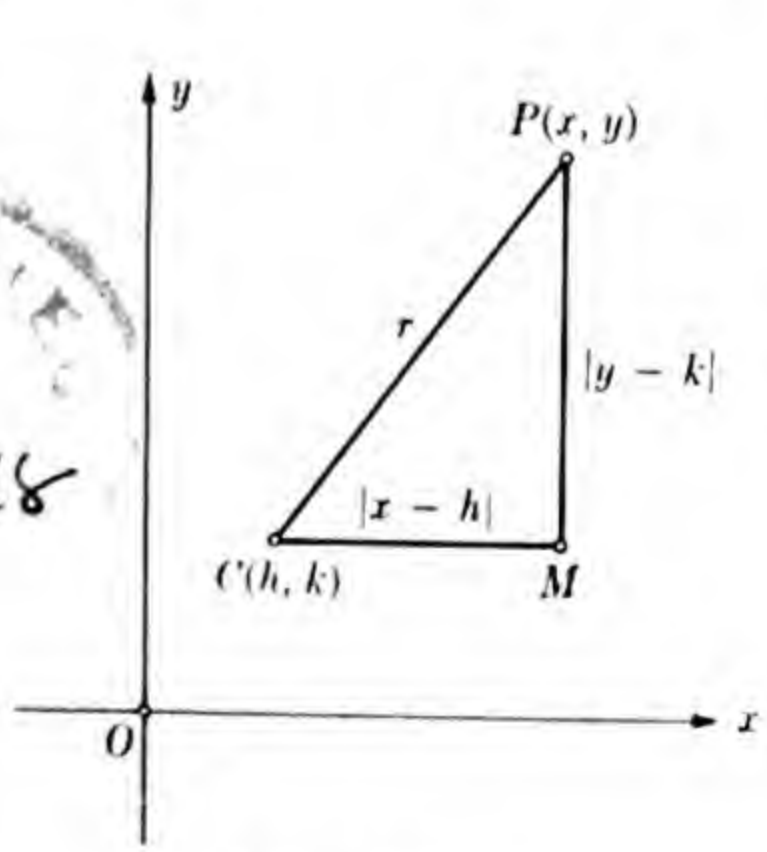


FIGURE 6–9

If Eqs. (6–23) and (6–24) are expanded, they take the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \tag{6–25}$$

where A and C are both positive constants and $A = C$ when the curve is a circle. Conversely, an equation in form (6–25) can be reduced to the form (6–24), by the method of completion of squares, in cases where that equation represents a circle or an ellipse.

EXAMPLE 6–15. Reduce the equation

$$x^2 + 2y^2 + 4x - 12y + 6 = 0$$

to the standard form (6–24) and locate the center and intercepts. Draw the ellipse and the corresponding auxiliary circle. Check the diagram by finding where the coordinate axes intersect the curve, using the original equation.

The procedure is as follows:

$$(x^2 + 4x \quad\quad) + 2(y^2 - 6y \quad\quad) = -6$$

$$(x^2 + 4x + 4) + 2(y^2 - 6y + 9) = -6 + 4 + 18 = 16$$

$$\frac{(x+2)^2}{16} + \frac{(y-3)^2}{8} = 1.$$

The auxiliary circle has equation $(x + 2)^2 + (y - 3)^2 = 16$. The center of the ellipse is $(-2, 3)$, and the ellipse cuts the line $y = 3$ where $x = -2 \pm 4$, and cuts the line $x = -2$, where $y = 3 \pm \sqrt{8}$ (Fig. 6–10).

If the given equation, with $y = 0$, is solved, it is noted that the intersections on the x-axis are not real. When the given equation is solved with $x = 0$, the y-intercepts are found to be $y = 3 \pm \sqrt{6}$.

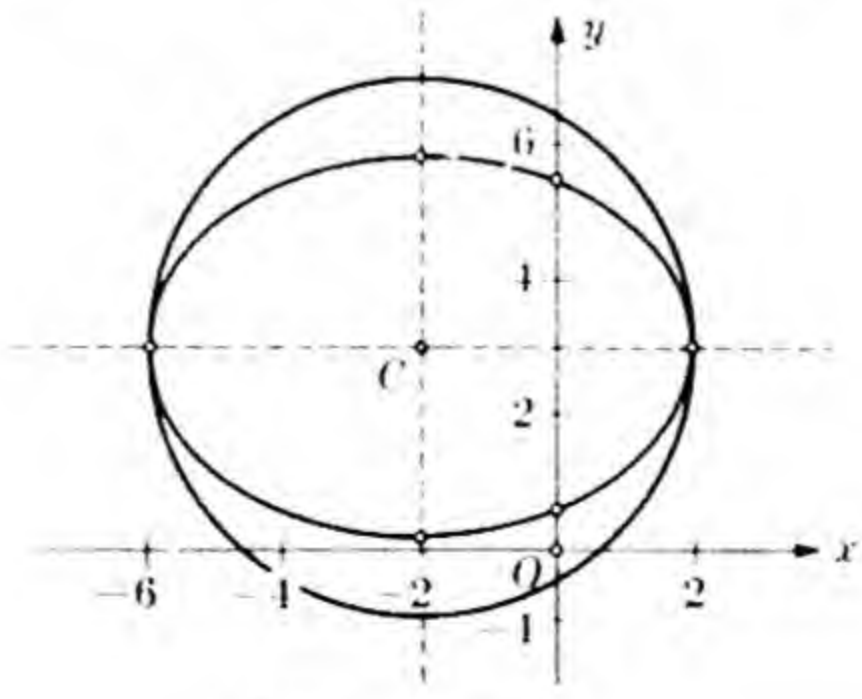


FIGURE 6–10

6–10 Hyperbolas. The graphs of the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \tag{6-26}$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad \text{or} \qquad y = \pm \frac{b}{a}\sqrt{x^2 + a^2}. \tag{6-27}$$

where a and b are positive constants, are called *hyperbolas.*

In sketching the curve corresponding to Eq. (6–26), it is noted that x cannot be less than a if y is to be real. For the hyperbola corresponding to Eq. (6–27), all values of x may be used but $|y| \geqq b$. The graphs of both of these curves are related to the graph of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

This equation corresponds to a pair of straight lines through the origin with slopes $\pm b/a$. When x is very large, y is also very large but x and y are so related that the hyperbolas come very close to these lines, called the *asymptotes* of the hyperbolas.

For many applied problems, the variables, x and y, are inversely proportional:

$$y = \frac{c}{x} \qquad \text{or} \qquad xy = c. \tag{6-28}$$

The curve corresponding to this equation is a special hyperbola called an *equilateral hyperbola.* Its asymptotes are the coordinate axis. If x is very large, y is very small; as $|x|$ increases, $|y|$ decreases toward zero. If y is very large, x is very small; as $|y|$ increases, $|x|$ decreases toward

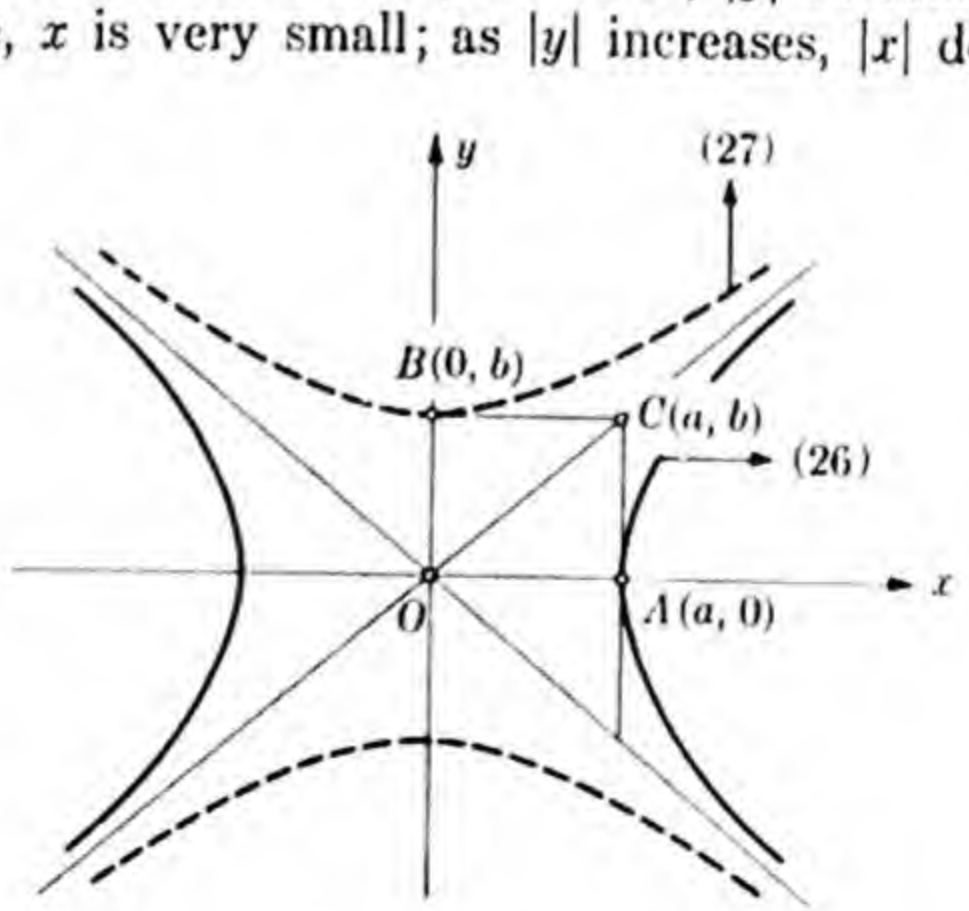


FIGURE 6–11

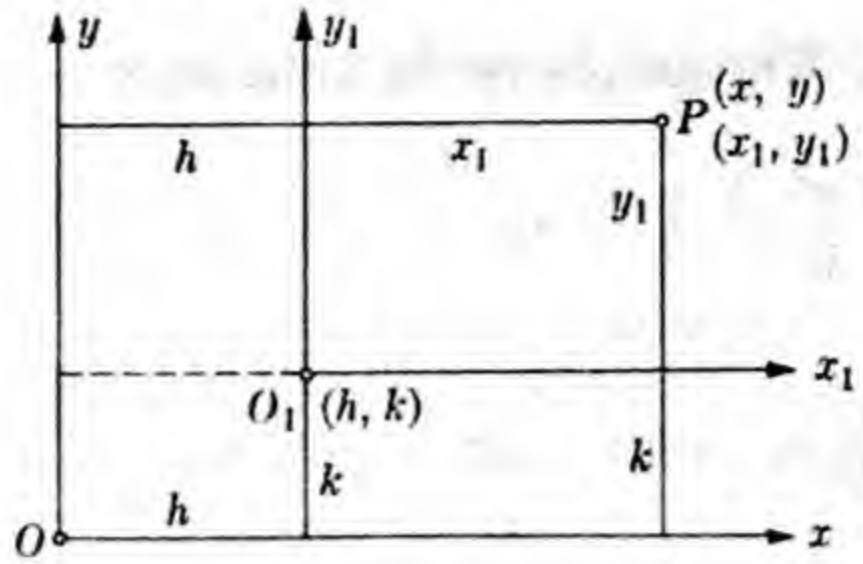


FIGURE 6–12

zero. The curve is quite easy to draw using $x = 0$ and $y = 0$ as guide lines. If $c > 0$, the curve appears in the first and third quadrants. If $c < 0$, the curve is in the second and fourth quadrants.

Translation of axes. The equations of hyperbolas may be written when the center is not at the origin. Before doing this, we discuss the topic of translation of axes. Let the coordinates of a point P be (x, y) and consider a new set of axes O_1x_1, O_1y_1, which are parallel to the origin axes. Let the original coordinates of O_1 be $x = h$, $y = k$. The point P has coordinates (x_1, y_1) when referred to the new axes, and the two systems of coordinates (Fig. 6–12) are related in terms of directed distances by the equations

$$x = x_1 + h, \qquad y = y_1 + k$$

or

$$x_1 = x - h, \qquad y_1 = y - k. \tag{6–29}$$

Equations (6–29) are known as the equations for translation of axes.

After completing the square, a translation of axes reduces the equation of an ellipse (see Eqs. (6–24) and (6–25)) to the form

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

The equation of the parabola with axes parallel to a coordinate axes ((Eqs. (6–10) and (6–11)) can be reduced to one of the forms

$$(y - k) = a(x - h)^2 \qquad \text{or} \qquad (x - h) = A(y - k)^2,$$

where (h, k) are the coordinates of the vertex V, and a translation of axes reduces them to

$$y_1 = ax_1^2 \qquad \text{or} \qquad x_1 = Ay_1^2.$$

A similar procedure can be used for the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where A and C have opposite signs. After completion of squares and a convenient translation of axes, this equation takes one of the two forms

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \pm 1.$$

The hyperbola which it represents is sketched from this form.

The equation

$$(x - h)(y - k) = c \tag{6–30}$$

becomes $x_1 y_1 = c$, when a translation of axes is made, and its graph is an equilateral hyperbola with asymptotes

$$x_1 = 0 \quad \text{and} \quad y_1 = 0 \qquad \text{or} \qquad x - h = 0 \quad \text{and} \quad y - k = 0.$$

If Eq. (6–30) is expanded, it takes the form

$$Bxy + Dx + Ey + F = 0, \qquad (B \neq 0), \tag{6–31}$$

and conversely, Eq. (6–31) can be reduced to form (6–30). The equation of the equilateral hyperbola can also be written as

$$y = \frac{c}{x - h} + k, \qquad \text{or} \qquad y = \frac{k(x - a)}{x - h}. \tag{6–32}$$

EXAMPLE 6–16. Discuss the graph of the equilateral hyperbola

$$y = \frac{2(x + 5)}{x + 3}.$$

(a) The horizontal asymptote is $y = 2$ and the vertical asymptote is $x = -3$. These lines are drawn as guide lines. The curve can be sketched from this information and the following points.

x	-3	∞	0	-5	5	-7
y	∞	2	$10/3$	0	$5/2$	1

(b) By clearing fractions and collecting terms, the given equation becomes

$$xy + 3y - 2x = 10.$$

It is then prepared for completing the factors:

$$(x - h)(y - k) = 10 + hk.$$

It is observed that to get the term $3y$, h must be -3; and for the term $-2x$, k must be 2. The product term, $hk = -6$, has been added to both

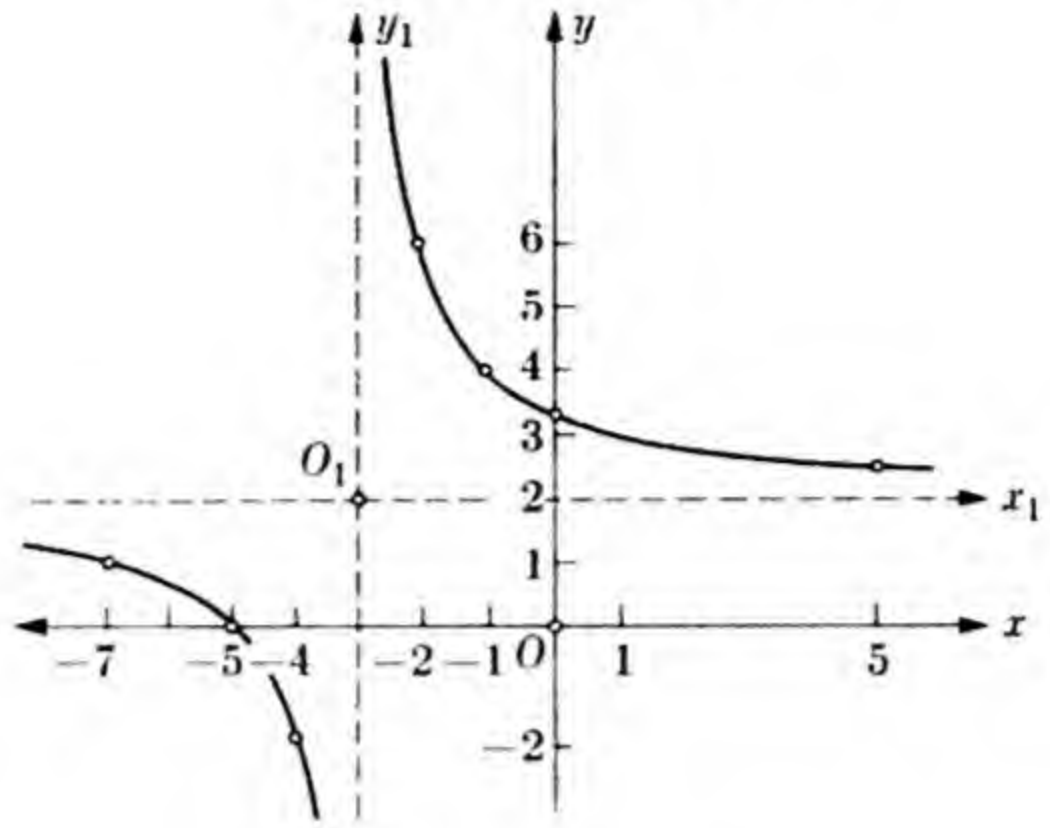


FIGURE 6–13

members and hence the process gives

$$(x + 3)(y - 2) = 10 - 6 = 4$$
$$x_1 y_1 = 4,$$

where $x_1 = x + 3$, $y_1 = y - 2$. From this last form additional points can be obtained and several such points are shown in the diagram (Fig. 6–13).

The branches of the equilateral hyperbola either rise consistently or fall consistently, thus giving a reason for its use in economic applications. Although the demand law, $px = c$, would not be realistic for very small value of x (large p) or very small value of p (large x), its simplicity has made it convenient for other values of p and x. The more general form

$$(x - h)(p - k) = c,$$

where h is negative, k and c are positive, is often realistic for commodities whose prices are not very sensitive to the quantity demanded. If in Example 6–16, y is interpretated as p, the equation

$$(p - 2)(x + 3) = 4, \qquad (x \geqq 0),$$

represents a demand function, where the highest price paid is 10/3 corresponding to $x = 0$, and where the average price tends to $p = 2$ as x increases.

PROBLEM SET 6–5

1. Sketch the following ellipses, using an auxiliary circle when convenient.

(a) $x^2 + y^2 = 16$ (b) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(c) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ (d) $2x^2 + y^2 = 1$

2. Sketch the following hyperbolas, showing the intercepts and asymptotes.

(a) $x^2 - y^2 = 1$ (b) $x^2 - y^2 = -1$

(c) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (d) $\frac{x^2}{16} - \frac{y^2}{25} = 1$

(e) $\frac{x^2}{16} - \frac{y^2}{9} = -1$ (f) $2x^2 - y^2 + 1 = 0$

3. Sketch the following equilateral hyperbolas, showing the asymptotes and such other points as are needed to complete the sketch.

(a) $xy = 9$ (b) $4xy = -9$

(c) $y = \frac{2}{x - 2}$ (d) $y = \frac{4}{2 - x}$

(e) $y = \frac{4}{x + 2}$ (f) $y = \frac{x}{4 - x}$, $(0 \leqq x \leqq 8)$

4. Simplify the following equations by means of a convenient translation of axes. If the curve is an ellipse, sketch the curve using an auxiliary circle. If the curve is an hyperbola, sketch the curve showing the asymptotes and a few other points. In each case, check the graph against intercepts on the original coordinate axes.

(a) $x^2 + y^2 + 4x - 6y = 0$

(b) $4x^2 + 4y^2 - 12x + 4y + 1 = 0$

(c) $x^2 + 4y^2 - 4x + 16y + 16 = 0$

(d) $3x^2 + 2y^2 - 12x - 12y + 24 = 0$

(e) $4x^2 + 12x + y - 27 = 0$

(f) $y^2 - x - 4y + 6 = 0$

(g) $x^2 - y^2 + 4x - 6y - 9 = 0$

(h) $x^2 - 4y^2 - 8x + 16y + 9 = 0$

(i) $3x^2 - 2y^2 - 12x - 12y - 12 = 0$

5. Simplify the following equations by means of a convenient translation of axes and sketch the curves. Show the asymptotes. Check the graph against intercepts on the original coordinate axes.

(a) $(x + 5)(y - 7) = 15$ (b) $(x - 40)(y - 30) = 300$

(c) $(x + 2)(y + 3) = 2$ (d) $xy - 6x + 3y - 18 = 0$

(e) $xy + 30y + 40x + 900 = 0$ (f) $2xy - 6x - 5y + 20 = 0$

6. Sketch the following demand or supply curves showing both asymptotes. Indicate the part of the curve that is significant.

(a) $(x + 2)p = 8$ (b) $px + 2p - 5x = 0$

(c) $(x + 2)(p - 2) = 10$ (d) $px - 5x + p - 1 = 0$

6–11 Intersection of a line with simple conics. The parabola, circle, ellipse, and hyperbola discussed in Sections 6–7, 6–8, 6–9 belong to a class of curves called *conics*. The equations were given for cases where the curves have relatively simple positions with reference to the coordinate axes, and these equations were special cases of the general equation of the second degree in the variables x and y. If the equation of a conic and straight line are given, their simultaneous solution may be found algebraically by eliminating one of the variables and then solving the resulting quadratic equation. Substitution in the linear equation then yields the value of the other variable. A graphical solution can be obtained by first sketching the conic and line in the same diagram and then reading coordinates of the points of intersection. The diagram may indicate that they have two, one, or no real points in common. The graphical procedure can also be used if both curves are conics. Except in special cases, the algebraic solution requires techniques which are beyond the scope of this book. The special cases of a conic and a line, and of two parabolas whose axes of symmetry are both parallel to the same coordinate axis, are now considered. When the curves represent economic situations in which both variables are positive, only positive solutions are retained.

EXAMPLE 6–17. Find, algebraically and graphically, the intersections of the line $2x + y = 8$ and (a) the hyperbola $xy = 8$; (b) the circle $x^2 + y^2 = 20$; (c) the parabola $y = 6x - x^2$.

(a) $y = 8 - 2x, \quad xy = 8$

$$8x - 2x^2 = 8$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)^2 = 0$$

$$x = 2, \qquad y = 8 - 4 = 4.$$

This indicates that the line is tangent at the point $T(2, 4)$ to the hyperbola (Fig. 6–14).

(b) $y = 8 - 2x, \qquad x^2 + y^2 = 20$

$$x^2 + 64 - 32x + 4x^2 = 20$$

$$5x^2 - 32x + 44 = 0$$

$$x = \frac{32 \pm \sqrt{1024 - 880}}{10} = \frac{32 \pm \sqrt{144}}{10} = \frac{32 \pm 12}{10} = 2 \text{ and } 4.4.$$

If $x = 2$, $y = 4$; if $x = 4.4$, $y = 8 - 8.8 = -0.8$. These results correspond to the points T and C in the diagram.

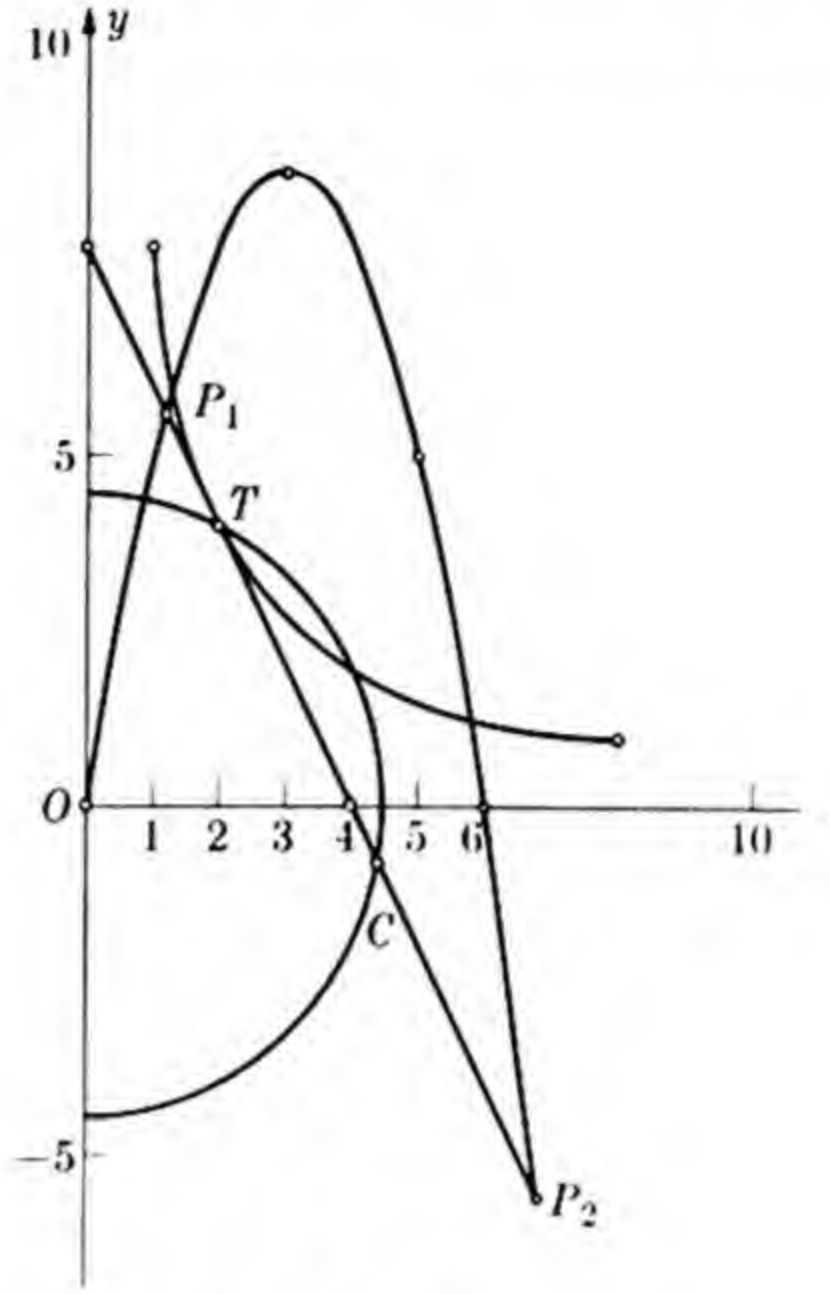


FIGURE 6–14

(c) $y = 8 - 2x, \qquad y = 6x - x^2$

$$8 - 2x = 6x - x^2$$

$$x^2 - 8x + (16) = -8 + (16) = 8$$

$$(x - 4)^2 = 8$$

$$x = 4 \pm \sqrt{8} = 6.83 \text{ and } 1.17.$$

If $x = 4 - \sqrt{8}$, $y = 2\sqrt{8} = 5.66$; if $x = 4 + \sqrt{8}$, $y = -2\sqrt{8} = -5.66$. These results correspond to the points P_1 and P_2 of Fig. 6–14, where only the diagram for $x \geqq 0$ is drawn. If the line corresponds to a demand law and the parabola to a supply law, then only the solution corresponding to $P_1(1.17, 5.66)$ is retained.

EXAMPLE 6–18. The total cost Q of producing x units of a commodity is given by $Q = \frac{1}{3}x^2 + \frac{2}{3}x + 4$, and the total revenue R received from the sale is $R = 6x - x^2$. For what values of x does the revenue exceed the cost? The profit F, defined as $R - Q$, is also a quadratic function of x. For what value of x is the profit a maximum?

The revenue curve and the cost curve, obtained by the methods previously discussed, are shown in Fig. 6–15. The revenue curve is drawn from its intercepts (0, 0), (6, 0) and the vertex (3, 9). The cost curve is drawn from its intercept (0, 4) and, since Q is monotonically increasing

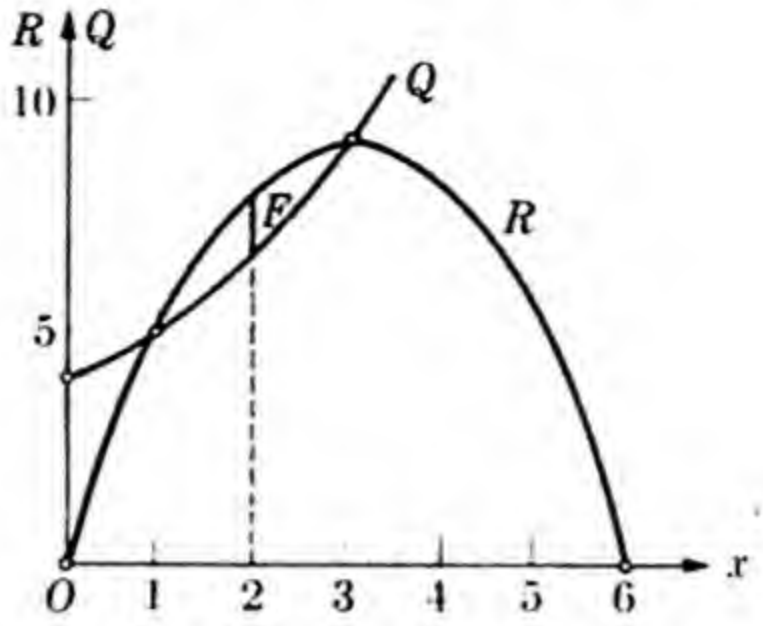


FIGURE 6–15

for $x \geqq 0$, several other points. The curves intersect at two points and the revenue exceeds the cost for the values of x between these points. The vertical distance between the curves represents the profit.

To find the points where the curves cross,

$$6x - x^2 = \tfrac{1}{3}x^2 + \tfrac{2}{3}x + 4$$

$$\tfrac{4}{3}x^2 - \tfrac{16}{3}x + 4 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0 \qquad \text{or} \qquad x = 1 \quad \text{and} \quad x = 3.$$

Hence the revenue exceeds the cost for $1 \leqq x \leqq 3$.

The profit F is given by

$$F = (6x - x^2) - (\tfrac{1}{3}x^2 + \tfrac{2}{3}x + 4)$$

$$= -\tfrac{4}{3}x^2 + \tfrac{16}{3}x - 4.$$

The method of completing the square yields

$$F = -4 - \tfrac{4}{3}(x^2 - 4x \quad)$$

$$= -4 + \tfrac{16}{3} - \tfrac{4}{3}(x^2 - 4x + 4)$$

$$= \tfrac{4}{3} - \tfrac{4}{3}(x - 2)^2.$$

This shows that maximum profit occurs when $x = 2$ and this profit has the value 4/3. All the above results agree with Fig. 6–15.

PROBLEM SET 6–6

1. Find the simultaneous solution of the following pairs of equations by graphical and algebraic methods.

(a) $x^2 + y^2 = 16$
$x + y = 4$

(b) $x^2 + y^2 = 16$
$x + 2y = 4$

(c) $x^2 + y^2 = 13$
$x - 2y + 4 = 0$

(d) $x^2 + y^2 = 13$
$3x + 2y = 13$

2. If the equations of two intersecting circles are given, $x^2 + y^2$ may be eliminated between them by subtraction to obtain the equation of the line which joins their points of intersection (common chord). The intersections of this line and the circle can then be found. Find the points of intersection of the following circles graphically and algebraically.

(a) $x^2 + y^2 = 16$
$x^2 + y^2 - 9x = 0$

(b) $x^2 + y^2 - 9y = 0$
$x^2 + y^2 - 9x = 0$

(c) $x^2 + y^2 = 25$
$x^2 + y^2 - 4x - 2y - 3 = 0$

(d) $x^2 + y^2 + 4x - 6y = 0$
$x^2 + y^2 - 6x + 6y - 2 = 0$

3. Find the intersections graphically and algebraically for the ellipse

$$x^2 + 4y^2 = 1$$

and each of the following lines.

(a) $x + 2y = 1$ (b) $x + 2y = 0$ (c) $x + 2y = -2$

4. Find the intersections graphically and algebraically for the following parabolas and lines.

(a) $y = 8x - x^2$
$x + 2y = 26$

(b) $y = 8x - x^2$
$x + 2y = 24$

(c) $y = x^2 - 5x + 4$
$x - 2y - 1 = 0$

(d) $y = x^2 - 5x + 4$
$x - 2y + 2 = 0$

5. Find the intersections graphically and algebraically for the following hyperbolas and lines.

(a) $4x^2 - 9y^2 = 32$
$2x + 3y = 12$

(b) $4x^2 - 9y^2 = 32$
$2x - y = 8$

(c) $xy = 8$
$2x + y = 10$

(d) $xy = 8$
$-2x + y = 10$

6. Determine the market equilibrium price and quantity, geometrically and algebraically, for the following demand and supply laws.

	DEMAND	SUPPLY
(a)	$px = 16$	$p = \frac{2}{3} + \frac{1}{3}x, (2 \leqq x \leqq 8)$
(b)	$px = 25$	$p = 3 + x, (2 \leqq x \leqq 10)$
(c)	$p = 30 - 4x$	$p = 2x^2 + 4x + 6$
(d)	$p = 30 - 3x$	$p = 2x^2 + 4x + 6$
(e)	$(x + 10)(p + 12) = 200$	$2x = 2p - 3$
(f)	$p(x + 6) = 120$	$p = 6 + \frac{1}{3}x$

7. Find the intersections for the following pairs of parabolas. Check graphically.

(a) $y = 6x - x^2$, $y = x^2 - 4x$

(b) $y = 5x - x^2$, $y = x^2 - 5x + 4$

(c) $x = y^2 - 2y$, $x = y^2 + 4y$

(d) $y = \sqrt{8 - 2x}$, $y = \sqrt{8 + 4x} - 2$

8. The total cost Q of producing x units of a commodity is given by $Q = x^2 + 2x$, and the total revenue R received from the sale is $R = 12x - 4x^2$. For what values of x does the revenue exceed the cost? The profit $F = R - Q$ is a quadratic function of x. For what value of x is the profit a maximum? Include a diagram.

9. Solve problem 8 with $R = 8x - x^2$ and $Q = \frac{1}{12}x^2 + \frac{5}{12}x + \frac{13}{2}$.

10. If the demand law is $p(x + 2) = 16$ and the supply law is $p = 1 + x^2/4$, the equilibrium quantity and price are difficult to find by algebraic means, but a good estimate can be determined graphically. Find such an estimate.

CHAPTER 7

LAWS OF ALGEBRA. EQUALITY AND INEQUALITY

7–1 Introduction. In this chapter we seek to develop a better understanding of the real number system rather than to develop techniques and skills in the fundamental operations of algebra or their use in the solution of equations. This requires some reconsideration of ideas already established on an intuitive basis but which now are discussed from a more theoretical point of view. Understanding of this material may be tested by supplying proofs for the theorems that are preceded by an asterisk.

This chapter also discusses inequalities involving one variable. The theory, techniques, and skills developed here are extended to inequalities with two and three variables in the next chapter, especially in Sections 8–5 and 8–6.

The concept of "real number" has evolved over many centuries and is one that must be developed in the sense that it is taken as one of the primitive undefined terms. Real numbers are represented by a variety of symbols such as $0, 1, 2, \ldots, 3/7, \ldots, 2.38, \ldots, \sqrt{3}, \ldots, \pi \ldots$ or by symbols such as $a, b, c, \ldots, x, y, z, \ldots$. Elliptically, these symbols are usually referred to as numbers, although they are really symbols which stand for numbers. This convention is followed here.

In order to develop the real number system from a logical basis, certain entities called *real numbers*, certain relationships connecting real numbers called *equal*, *greater than* or *less than*, and the two operations *addition* and *multiplication* are taken as undefined, subject to explicitly stated *Axioms*. These axioms are statements accepted without proof. Additional quantities are defined in terms of these primitive terms and/or previously defined terms. Certain statements called *Theorems* are proved in accordance with the usual laws of logic from these axioms and previously proved theorems.

Various axiom systems have been given for the real numbers. The system used here is a compilation and rearrangement of systems that can be found in many sources.† No claim is made that it is the best system for

---

† See Whyburn and Daus, *Algebra for College Students*, Prentice-Hall, 1955; Department of Mathematics, University of Kansas, *Universal Mathematics*, Part I, 1954; Daus and Whyburn, *Introduction to Mathematical Analysis*, Addison-Wesley, 1958; especially pp. 35–38. The authors were recently influenced in this selection by the work of the School Mathematics Study group, office at Yale University. See *Studies in Mathematics*, Volume II, *Euclidean Geometry Based on Ruler and Protractor Axioms*, by Curtis, Daus, and Walker.

all purposes. The axiom system for real numbers is conveniently divided into three sets: (1) the axioms for a *number field* concerning *equality*, *addition*, and *multiplication;* (2) the axioms of *order* concerning *less than* and *greater than* or the equivalent notions of *positive* and *negative numbers;* (3) an axiom of *continuity*, or the *least upper bound* axiom, which is needed to insure the completeness of the real number system. In brief, the real number system is the set of entities which satisfy the system of axioms (and their consequences) presently to be discussed. As the discussion progresses, it is well to keep in mind the set of all infinite (and terminating) decimal fractions as a model of the real number system.

7–2 Equality. The following axioms concerning real numbers a, b, c and the relationship of $=$ are assumed.

AXIOM E1. *Equality is reflexive.*

$$a = a.$$

AXIOM E2. *Equality is symmetric.*

If $a = b$, *then* $b = a$.

AXIOM E3. *Equality is transitive.*

If $a = b$ *and* $b = c$, *then* $a = c$.

AXIOM E4. THE RULE OF SUBSTITUTION.

If a occurs in any algebraic expression and if $a = b$, *then b may be substituted for a to give an algebraic expression which is equivalent to the original expression.*

7–3 Addition and subtraction. The operation of adding two real numbers is indicated by the symbol $+$. The following axioms which involve addition are assumed.

AXIOM A1. CLOSURE. *If a and b are real numbers, there is a unique real number* $a + b$, *called their sum.*

AXIOM A2. *Addition is commutative.*

If a and b are real numbers, then $a + b = b + a$.

AXIOM A3. *Addition is associative.*

If a, b, c are real numbers, then

$$(a + b) + c = a + (b + c) = a + b + c.$$

The parentheses are used to indicate that the operation within them is to be performed first. Since the result is independent of the order in which the additions are made, the parentheses may be omitted.

AXIOM A4. ZERO. *For any real number a, there exists a unique real number, called zero and represented by the symbol* 0, *such that*

$$a + 0 = a.$$

* It follows that†

$$a + 0 = 0 + a = a.$$

AXIOM A5. ADDITIVE INVERSE. *For each real number a, there exists a unique real number, called the additive inverse of a and represented by the symbol* $(-a)$, *such that*

$$a + (-a) = 0.$$

THEOREM: $(-(-a)) = a$.

Proof. In view of the commutative law of addition, $(-a) + a = 0$, so that a is the additive inverse of $(-a)$, and the Theorem follows from the definition of the additive inverse.

THEOREM 7-1. *For any real numbers a, b, c, d, if* $a = b$ *and* $c = d$, *then* $a + c = b + c$ *and* $a + c = b + d$.

Proof. Start with the sum $a + c$, and apply the rule of substitution, replacing a by b:

$$a + c = b + c.$$

A second substitution gives

$$a + c = b + d.$$

The converse is now considered.

THEOREM 7-2. LAW OF CANCELLATION FOR ADDITION.

If $a + c = b + c$, *then* $a = b$.

The proof is presented in two-column form.

$(a + c) + (-c) = (b + c) + (-c)$	Substitution
$a + [c + (-c)] = b + [c + (-c)]$	Associative
$a + 0 = b + 0$	Additive inverse
$a = b$	Definition of zero.

† The asterisk is used in this chapter to indicate that the proof has been omitted and should be supplied by the reader.

COROLLARY. *If* $a + c = b + d$ *and* $c = d$, *then* $a = b$.

DEFINITION OF SUBTRACTION. If x, a, b are real numbers such that $x + a = b$, then x is the result of subtracting a from b. The operation of subtraction is represented by the minus sign, $-$.

$$x + a = b \quad \text{means} \quad x = b - a.$$

THEOREM 7–3.* *For any pair of real numbers* a *and* b *there is a unique real number* x *such that* $x + a = b$, *namely,* $x = b + (-a)$.

Because of its importance this was proved as Theorem 1–1. The reader should supply the details of the proof which requires a few changes in references.

COROLLARY. *If* a *and* b *are real numbers,*

$$b - a = b + (-a).$$

Proof. This follows from the definition of subtraction and the uniqueness of x in Theorem 7–3.

Remark. This Corollary justifies use of the symbol $-$ to represent both the operations of subtraction and the formation of the additive inverse of a number; $(-a)$ is also called the "negative of a" or "minus a." When no misunderstanding can occur, $-a$ is written instead of $(-a)$ and $-(-a)$ is written instead of $(-(-a))$.

This Corollary is the basis for the rules of sign when removing or inserting parentheses.

THEOREM 7–4.* *For each real* a

$$a - a = 0 \quad \textit{and} \quad a = a - 0.$$

THEOREM 7–5. *If* a *and* b *are real numbers, then*

$$-(a + b) = -a - b \quad \textit{and} \quad -(a - b) = b - a.$$

Proof: Let $x = -(a + b) = 0 - (a + b)$.

Then $x + a + b = 0$

and $x = (-a) + (-b) = -a - b$.

Let $y = -(a - b) = 0 - (a - b)$.

Then $y + (a - b) = 0$

$$y + [a + (-b)] = 0$$

$$y = (-a) - (-b) = b - a.$$

COROLLARY. *If a, b, c are real numbers,*

$$c - (a + b) = c - a - b \quad \textit{and} \quad c - (a - b) = c - a + b.$$

PROBLEM SET 7–1

(All proofs should be precise and with appropriate references.)

1. If zero is defined by $a + 0 = a$, prove $0 + a = a$.

2. Give and prove the subtraction facts which correspond to the definition of zero (Theorem 7–4).

3. Prove the Corollary to Theorem 7–2:

$$\text{If} \quad a + c = b + d \quad \text{and} \quad c = d, \quad \text{then} \quad a = b.$$

4. (a) Prove Theorem 7–3: The equation $x + a = b$ has one and only one solution. (b) Discuss the case when $b = 0$.

5. Prove that if $a = b$ and $c = d$, then $a - c = b - d$ and, conversely, if $a - c = b - d$ and $c = d$, then $a = b$.

6. Prove the Corollary to Theorem 7–5:

$$c - (a + b) = c - a - b \quad \text{and} \quad c - (a - b) = c - a + b.$$

7. If a, b, c are real numbers, prove

$$(a - b) - (c - b) = a - c.$$

7–4 Multiplication. The operation of multiplying two real numbers is indicated by the use of symbol $\times$, the center dot $\cdot$, or by placing the numbers in juxtaposition. The following axioms concerning multiplication are assumed.

AXIOM M1. CLOSURE. *If a and b are real numbers there is a unique real number ab, called their product.*

AXIOM M2. *Multiplication is commutative.*

If a and b are real numbers, then $ab = ba$.

AXIOM M3. *Multiplication is associative.*

If a, b, c are real numbers, then

$$(ab)c = a(bc) = abc.$$

The parentheses are used to indicate that the operation within them is to be performed first. Since the result is independent of the order in which the multiplications are made, the parentheses may be omitted.

AXIOM M4. ONE. *For any real number a, there exists a unique real number, called one and represented by the symbol* 1, *such that*

$$a \cdot 1 = a.$$

COROLLARY. $$a \cdot 1 = 1 \cdot a = a.$$

AXIOM M5. MULTIPLICATIVE INVERSE. *For each real number a different from* 0, *there exists a unique real number, called the multiplicative inverse or reciprocal of a and represented by the symbol* $1/a$, *such that*

$$a(1/a) = 1.$$

COROLLARY. $$1/(1/a) = a.$$

Proof. In view of the commutative law of multiplication

$$(1/a)a = 1,$$

so that a is the multiplicative inverse of $1/a$, and the stated corollary follows from the definition of the multiplicative inverse.

THEOREM 7–6.* *For any real numbers a, b, c, d, if* $a = b$ *and* $c = d$, *then* $ac = bc$ *and* $ac = bd$.

The proof of this theorem is analogous to the proof of Theorem 7–1 and is left for the reader. There is a close analogy between the discussion for addition and that for multiplication.

Since the reciprocal of 0 was not defined, this must be taken into account when the converse theorem is stated and proved. The proof is analogous to the proof of Theorem 7–2.

THEOREM 7–7. LAW OF CANCELLATION FOR MULTIPLICATION.

If $ac = bc$, $(c \neq 0)$, *then* $a = b$.

The proof is presented in two-column form.

$(ac)(1/c) = (bc)(1/c)$	Substitution (or Theorem 7–6)
$a[c(1/c)] = b[c(1/c)]$	Associative
$a \cdot 1 = b \cdot 1$	Multiplicative inverse
$a = b$	Definition of 1.

COROLLARY. *If* $ac = bd$ *and* $c = d \neq 0$, *then* $a = b$.

DEFINITION OF DIVISION. If x, a, b are real numbers such that $xa = b$, then x is the result of dividing b by a. The operation of division is represented by the solidus, /, or a horizontal bar, ——.

$$xa = b \quad \text{means} \quad x = b/a \quad \text{or} \quad \frac{b}{a}.$$

THEOREM 7-8.* *For any pair of real numbers a and b, with $a \neq 0$, there is a unique real number x such that $xa = b$, namely, $x = b(1/a)$.*

Proof. $b(1/a)$ is a solution of the equation $xa = b$ because

$$[b(1/a)]a = b[(1/a)a] \qquad \text{Why?}$$
$$= b1 = b \qquad \text{Why?}$$

The uniqueness follows from the law of cancellation for multiplication applied to $xa = b = ya$. The details are left to the reader. (Problem 5 of Set 7-2.)

COROLLARY. *If a and b are real numbers, with $a \neq 0$,*

$$\frac{b}{a} = b\left(\frac{1}{a}\right).$$

Proof. This follows from the definition of division and the uniqueness of x in Theorem 7-8.

Remark. This Corollary justifies use of the symbol / to represent both the operation of division and the formation of the reciprocal of a number. The fundamental laws of division and operations with fractions are reduced to multiplications. They were discussed in Chapter 3 and are not repeated here.

THEOREM 7-9.* *For each real number a different from 0, $a/a = 1$, and for all a, $a/1 = a$.*

The next axiom is the fundamental law connecting the operations of addition and multiplication. This axiom is the basis for the discussion of the multiplicative properties of 0 and (-1), which were defined in terms of addition.

AXIOM M6. *Multiplication is distributive with respect to addition.*

For any real numbers a, b, c

$$(a + b)c = ac + bc.$$

COROLLARY. $c(a + b) = ac + bc$.

THEOREM 7-10.* *For any a, $a0 = 0$.*

COROLLARY. $0a = 0$.

THEOREM 7-11.* *If $ab = 0$, then either $a = 0$ or $b = 0$.*

Because of their importance, these two theorems were proved earlier as Theorem 2–1 and Theorem 2–2. The reader should repeat the details of the proofs, making a few needed changes in references.

The numbers 0 and 1 were defined by means of different operations and there is no à priori reason why they might not be the same.

THEOREM 7–12. $0 \neq 1$.

Proof. Suppose $a \neq 0$. If $0 = 1$, then

$$a0 = a1 \qquad \text{Rule of substitution}$$

$$0 = a,$$

because of Theorem 7–10, the definition of 1, and the laws of equality. This is impossible and hence $0 \neq 1$.

Theorems 7–10 and 7–12 explain why the reciprocal of 0, and hence division by 0, is not defined. The reciprocal of 0 would be the solution of the equation $x0 = 1$, but this equation has no solution since $x0 = 0 \neq 1$.

THEOREM 7–13.* *For any* a, $(-1)a = -a$.

The steps of the proof are given below, and it is left to the reader to supply appropriate reasons. Start with the expression $a + (-1)a$. Then

$$\begin{aligned} a + (-1)a &= 1a + (-1)a && \text{Why?} \\ &= [1 + (-1)]a && \text{Why?} \\ &= 0 \cdot a && \text{Why?} \\ &= 0 && \text{Why?} \end{aligned}$$

Therefore

$$(-1)a = (-a) = -a. \qquad \text{Why?}$$

This is a third interpretation of the symbol $-a$ and is the basis for the proof of the following theorems (see proofs of Theorem 2–5).

THEOREM 7–14:* *If a and b are real numbers, then*

$$a(-b) = (-a)b = -ab; \qquad (-a)(-b) = ab.$$

COROLLARY. $(-1)(1) = -1; \qquad (-1)(-1) = 1.$

PROBLEM SET 7–2

(All proofs should be precise and with appropriate references.)

1. If 1 is defined by $a1 = a$, prove $1a = a$.
2. Give and prove the two division facts which correspond to the definition of 1.

3. Prove Theorem 7-6: If $a = b$ and $c = d$, then $ac = bc$ and $ac = bd$.

4. Prove the Corollary to Theorem 7-7: If $ac = bd$ and $c = d \neq 0$, then $a = b$.

5. (a) Complete the details of the proof of Theorem 7-8: The equation $xa = b$, $(a \neq 0)$, has one and only one solution. (b) Discuss the case corresponding to $b = 0$.

6. Prove Theorems 7-10 and 7-11: If a and b are real numbers, then $ab = 0$ if and only if $a = 0$ or $b = 0$.

7. Supply the omitted reasons in the proof of Theorem 7-13: For all a, $(-1)a = -a$.

8. Prove Theorem 7-14: $a(-b) = (-a)b = -ab$ and $(-a)(-b) = ab$.

9. Use Theorem 7-13, $(-1)a = -a$, and the distributive law to prove $-(a + b) = -a - b$ and $-(a - b) = b - a$. (See Theorem 2-5; compare this method with that used to prove Theorem 7-5.)

10. Prove that if $a/c = b/d$ and $c = d \neq 0$, then $a = b$.

7-5 Order and inequality. The order relationship between two real numbers that are not equal is indicated by the symbols $>$ and $<$, which are read: "greater than" and "less than," respectively.

AXIOM O1. TRICHOTOMY. *If a is a real number, then one and only one of the following possibilities holds:*

$$\text{(i) } a > 0, \qquad \text{(ii) } a = 0, \qquad \text{(iii) } -a > 0.$$

DEFINITION 7-1. If $a > 0$, then a is *positive*.

AXIOM O2. *If a and b are positive, then $a + b$ is positive.*

AXIOM O3. *If a and b are positive, then ab is positive.*

Expressed in symbols these axioms are:

If $a > 0$ *and* $b > 0$, *then* $a + b > 0$ *and* $ab > 0$.

DEFINITION 7-2. $a > b$ (*a is greater than b*) if and only if $a - b$ is positive, that is, $a - b > 0$.

DEFINITION 7-3. $a < b$ (*a is less than b*) if and only if $b - a$ is positive, that is, $b - a > 0$.

The inequality $b - a > 0$ (in accord with Definition 7-2) implies $b > a$, and hence $a < b$ and $b > a$ are equivalent statements. Similarly, $a > b$ and $b < a$ are equivalent statements. The inequality may be read from left to right or right to left. The theorems of this section and the next section are usually stated in terms of "greater than," but the reader can readily translate any such statement into one involving "less than."

THEOREM 7–15. GENERAL LAW OF TRICHOTOMY. *If a and b are any real numbers, then one and only one of the following possibilities holds:*

$$\text{(i) } a > b, \qquad \text{(i) } a = b, \qquad \text{(iii) } a < b.$$

Proof. In accord with the Axiom of Trichotomy, either (i) $a - b > 0$, (ii) $a - b = 0$, (iii) $-(a - b) > 0$. By Definition 7–2, the first of these possibilities implies $a > b$; the second possibility implies $a = b$; and since $-(a - b) = b - a$, the third possibility, by Definition 7–3 implies $b - a > 0$ or $a < b$, and the proof is complete.

DEFINITION 7–4. If $-a > 0$, then a is *negative*.

THEOREM 7–16. *The real number a is negative ($-a > 0$), if and only if $a < 0$.*

Proof. This follows from Definition 7–3 for the case $b = 0$: $a < 0$ if and only if $0 - a = -a > 0$, in which case a is negative.

COROLLARY 1. *If a is positive and b is negative, then ab is negative.*

Proof. $a > 0$ and $-b > 0$ yield $-ab > 0$ (Theorem 7–14 and Axiom O3). Hence, by Theorem 7–16, $ab < 0$ or ab is negative.

COROLLARY 2. *If a is negative and b is negative, then ab is positive.*

THEOREM 7–17.* *The quotient* (i) *of two positive numbers is positive;* (ii) *of a positive number and a negative number is negative, and* (iii) *of two negative numbers is positive.*

Proof of (i). $a/b = c$, $(b \neq 0)$, if and only if $a = bc$ (Definition of division). Assume a and b are positive. If $c = 0$, then $a = 0$, which is impossible; if c were negative, then (Theorem 7–16, Cor. 1) a would be negative, which is impossible. Hence c is positive (Axiom of Trichotomy).

The proofs for (ii) and (iii) are similar.

7–6 Algebraic operations with inequality. A very useful statement which is equivalent to the definitions of greater than and less than is the following:

THEOREM 7–18. *For any real numbers a and b,*

(1) *$a > b$ if and only if there exists a positive number x such that*

$$a = b + x;$$

(2) *$a < b$ if and only if there exists a positive number y such that*

$$a + y = b.$$

Proof of (1). $a > b$ if and only if $a - b = x$, where x is a positive number. This is equivalent to $a = b + x$.

Proof of (2). $a < b$ if and only if $b - a = y$, where y is a positive number. This is equivalent to $a + y = b$.

Proofs of the theorems that follow may be made to depend upon Theorem 7–18 or Definitions 7–2 and 7–3. Both methods are illustrated.

THEOREM 7–19. TRANSITIVE. *If* $a > b$ *and* $b > c$, *then* $a > c$.

Proof. $a = b + x$ and $b = c + y$, where x and y are positive. By substitution, $a = c + (x + y)$, where $x + y$ is positive (Axiom O2). Hence $a > c$.

The compound statement $a > b$ and $b > c$ is written in abbreviated form as $a > b > c$.

COROLLARY. *If* $a < b$ *and* $b < c$, *then* $a < c$.
The compound statement of the hypothesis is written as $a < b < c$.

DEFINITION 7–5. If either $a > b > c$ or $a < b < c$, then b is *between* a and c.

THEOREM 7–20. ADDITION. *If* $a > b$, *then* $a + c > b + c$.

Proof. From the definition of ">" (Def. 7–2),

$$(a + c) - (b + c) = a - b > 0 \qquad \text{or} \qquad a + c > b + c.$$

THEOREM 7–21. MULTIPLICATION. (i) *If* $a > b$ *and* $c > 0$, *then* $ac > bc$; *and* (ii) *if* $a > b$ *and* $c < 0$, *then* $ac < bc$.

Proof. (i) If $a > b$ and $c > 0$, that is, if $a - b$ and c are positive, then

$$ac - bc = (a - b)c > 0, \qquad \text{or} \qquad ac > bc,$$

since the product of two positive numbers is positive.

(ii) If $a > b$ and $c < 0$, that is, if $a - b$ and $(-c)$ are positive, then

$$bc - ac = -(ac - bc) = (-c)(a - b) > 0,$$

since the product of two positive numbers is positive. Hence $ac < bc$ by Definition 7–3.

THEOREM 7–22. NEGATIVE. *If* $a > b$, *then* $-a < -b$.

Proof. $a = b + y$, where y is positive. Hence

$$-a = -(b + y) = -b - y$$

$$-a + y = -b.$$

Therefore, $-a < -b$ by Theorem 7–18.

THEOREM 7–23. MULTIPLICATION OF INEQUALITIES. *If* $a > b$ *and* $c > d$, *and if* b *and* d *are positive, then* $ac > bd$.

Proof.

$$\begin{array}{ll} a = b + x & x \text{ positive} \\ c = d + y, & y \text{ positive} \\ \hline ac = bd + by + dx + xy. & \end{array}$$

From Axioms O3 and O2 it follows that by, dx, xy and their sums are positive. Hence $ac > bd$.

The limitation that b and d be positive is essential. Otherwise one of the products by or dx would be negative and no conclusion could be drawn about the sum $by + dx + xy$. If b and d are not both positive, the conclusion might be false. For example, $3 > -2$ and $2 > -1$ and it is true that $6 > (-2)(-1) = 2$, but also $3 > -2$ and $2 > -4$ and it is false that $6 > (-2)(-4) = 8$.

COROLLARY. If $a > b > 0$, *then* $a^2 > b^2$.

If, however, $b < 0$, no such conclusion can be drawn. For example, $2 > -3$, but $4 \not> 9$.

THEOREM 7–24. SQUARE. *If* $a \neq 0$, *then* $a^2 > 0$.

Proof. Either $a > 0$, in which case $a^2 > 0$ by Axiom O3, or $-a > 0$, in which case $a^2 = (-a)^2 > 0$ by Theorem 7–14 and the same axiom.

In Theorem 7–12 it was shown that $1 \neq 0$; it is now possible to prove more:

COROLLARY 1. $1 > 0$

Proof. $1 = 1^2 \neq 0$. Hence the theorem can be applied, showing $1 > 0$.

COROLLARY 2. $-1 < 0$.

COROLLARY 3. *If* a *and* b *are different real numbers, then* $a^2 + b^2 > 2ab$.

Proof.

Start with	$(a - b)^2 > 0$	Theorem 7–24.
Then	$a^2 - 2ab + b^2 > 0$	Multiplication
and	$a^2 + b^2 > 2ab$	Theorem 7–20.

PROBLEM SET 7–3

(All proofs should be precise with appropriate references.)

1. Prove Corollary 2, Theorem 7–16: If a and b are both negative, then ab is positive.

2. Prove the second and third parts of Theorem 7–17: the quotient of a positive number and a negative number is negative; the quotient of two negative numbers is positive.

3. Prove the Corollary to Theorem 7-19: If $a < b$ and $b < c$, then $a < c$.

4. Theorem 7-19 states: "If $a > b$ and $b > c$, then $a > c$." Is the converse statement: "If $a > b$ and $a > c$, then $b > c$" true or false? Explain.

5. Theorem 7-20 states: "If $a > b$, then $a + c > b + c$." Is the converse true? Prove your answer.

6. Prove the following converse of Theorem 7-21: If $ac > bc$ and $c > 0$, then $a > b$.

7. Prove the theorem corresponding to Theorem 7-21(i) stated in terms of "less than."

8. Prove Theorem 7-21(ii): If $a > b$ and $c < 0$, then $ac < bc$, using Theorem 7-18.

9. Prove the theorem corresponding to Theorem 7-23 (Multiplication of Inequalities) stated in terms of "less than." Give several numerical examples to illustrate the necessity of the limitations on the numbers involved.

10. Prove Theorem 7-22: If $a > b$, then $-a < -b$ on the basis of Theorem 7-24, Corollary 2 and Theorem 7-21.

11. Prove the Corollary: If $a > b > 0$, then $a^2 > b^2$. If $a^2 > b^2$, what can be said about the relation between a and b?

12. If $a < b$, prove

$$a < \frac{a+b}{2} < b, \qquad \text{and} \qquad a < \frac{2a+b}{3} < \frac{a+2b}{3} < b.$$

7-7 Linear inequalities. The solution of inequalities is similar to the solution of equalities with the theorems concerning addition and multiplication, Theorems 7-20, 7-21, playing essential roles. The main difference is that for inequalities, the domain of the solution set is one or more intervals.

Inequalities of the form

$$ax + b \geqq c, \qquad (a > 0)$$

have the solution

$$x \geqq \frac{c-b}{a}, \text{ (Theorems 7-20 and 7-21(i))}.$$

If, however, a is negative, the linear inequality

$$ax + b \geqq c, \qquad (a < 0)$$

has the solution

$$x \leqq \frac{c-b}{a} \text{ (Theorems 7-20 and 7-21(ii))}.$$

If the original inequality signs are reversed, so are the final ones.

To solve linear inequalities which involve absolute values, the following theorems are used.

THEOREM 7–25. *The inequality*

$$|ax + b| > c, \qquad (c > 0)$$

is satisfied if and only if either

$$ax + b > c \qquad or \qquad -(ax + b) > c.$$

Proof. If $ax + b$ is positive, then $|ax + b| = ax + b$, and $|ax + b| > c$ if and only if $ax + b > c$. If $ax + b$ is negative, then $|ax + b| = -(ax + b)$ and $|ax + b| > c$ if and only if $-(ax + b) > c$. $ax + b$ cannot be zero and satisfy the given inequality, and the proof is complete.

THEOREM 7–26. *The inequality*

$$|ax + b| < c, \qquad (c > 0)$$

is satisfied if and only if both

$$ax + b < c \qquad and \qquad -(ax + b) < c,$$

that is, if

$$-c < (ax + b) < c.$$

Proof. If $ax + b$ is positive, $|ax + b| = ax + b$ and $|ax + b| < c$ if and only if $0 < ax + b < c$. If $ax + b$ is negative, $|ax + b| = -(ax + b)$ and $|ax + b| < c$ if and only if $-(ax + b) < c$, which is equivalent to $-c < (ax + b)$. If $ax + b$ is zero, it lies between $-c$ and c, and the proof is complete.

Theorems 7–25 and 7–26 can be given a convenient geometric interpretation on the number scale. If x_1 and x_2 are the solutions of $|ax + b| = c$, then all points between x_1 and x_2 satisfy $|ax + b| < c$ and all other points except x_1 and x_2 satisfy $|ax + b| > c$. This gives a convenient way of solving the inequalities (Fig. 7–1).

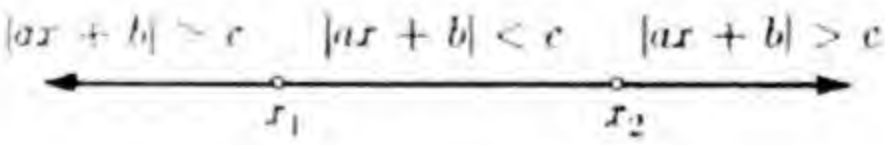


FIGURE 7–1

EXAMPLE 7–1. Solve the inequalities

(a) $4x + 3 \geqq 5$; (b) $|4x + 3| \geqq 5$; (c) $|4x + 3| < 5$.

(a) $4x + 3 \geqq 5$

$4x \geqq 2$

$x \geqq \frac{1}{2}$.

(b) $|4x + 3| \geqq 5$ is satisfied if either

$$4x + 3 \geqq 5 \quad \text{or} \quad -4x - 3 \geqq 5.$$

The first of these was solved above to find $x \geqq \frac{1}{2}$. For the second,

$$-4x \geqq 8$$
$$x \leqq -2.$$

Hence any x such that $x \geqq \frac{1}{2}$ or such that $x \leqq -2$ is a solution.

(c) $|4x + 3| < 5$ is replaced by

$$-5 < 4x + 3 < 5$$
$$-8 < 4x < 2$$
$$-2 < x < \tfrac{1}{2}.$$

Hence x is between -2 and $\frac{1}{2}$, which agrees with the previous result. See Fig. 7-2.

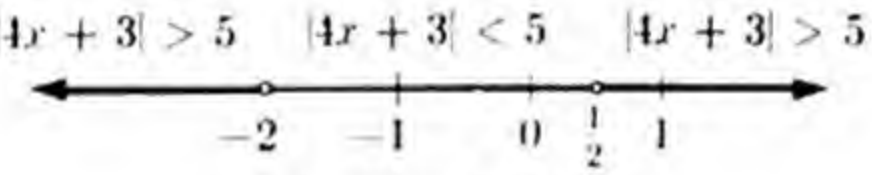


FIGURE 7-2

An alternative method of solution would be to solve $|4x + 3| = 5$ as follows:

$$|4x + 3| = \sqrt{(4x + 3)^2} = 5$$
$$(4x + 3)^2 = 25$$
$$16x^2 + 24x - 16 = 0$$
$$2x^2 + 3x - 2 = 0.$$

Since the coefficients in the original inequality were rational, the solution of this equation must be rational. Hence it can be factored over the integers. It is readily verified that the factored form is

$$(x + 2)(2x - 1) = 0,$$

which gives $x = \frac{1}{2}$ and $x = -2$. Hence the solution of $|4x + 3| < 5$ is the set of numbers between -2 and $\frac{1}{2}$, and the solution of $|4x + 3| > 5$ is the set of numbers which are greater than $\frac{1}{2}$ together with the set of numbers which are less than -2.

Inequalities that involve fractions are often reducible to linear inequalities by clearing of fractions while observing certain cautions. First, the values for which a denominator vanishes must be excluded. Second,

the common denominator used may be positive for some values of the variable and negative for other values of the variable. When clearing of fractions, the inequality is reversed for those values for which the common denominator is negative.

EXAMPLE 7–2. Solve the inequality

$$\frac{4}{x-2} < 2.$$

The inequality is meaningless for $x = 2$.

If $x > 2$, $x - 2$ is positive, and multiplication of the given inequality by $x - 2$ yields

$$4 < 2(x-2)$$
$$8 < 2x$$
$$x > 4.$$

The two inequalities are both satisfied by $x > 4$.

If $x < 2$, $x - 2$ is negative, and multiplication by $x - 2$ yields

$$4 > 2(x-2)$$
$$8 > 2x$$
$$x < 4.$$

The two inequalities are both satisfied by $x < 2$. Hence

$$\frac{4}{x-2} < 2$$

if $x < 2$ or if $x > 4$.

A graphical solution of the problem (Fig. 7–3) can be obtained by sketching the equilateral hyperbola $y = 4/(x - 2)$ and the line $y = 2$. The hyperbola may be sketched from its vertical asymptote $x = 2$, its horizontal asymptote $y = 0$, its intercept $x = 0$, $y = -2$, and a few other points obtained by substitution and symmetry. It is then possible to observe the values of x for which the hyperbola is below the line, namely,

$$x < 2 \quad \text{and} \quad x > 4.$$

The same diagram also shows that $[4/(x - 2)] > 2$ for $2 < x < 4$. The algebraic verification of this is left as an exercise.

EXAMPLE 7–3. Solve the inequality

$$\frac{x}{x+1} > 2$$

algebraically and graphically.

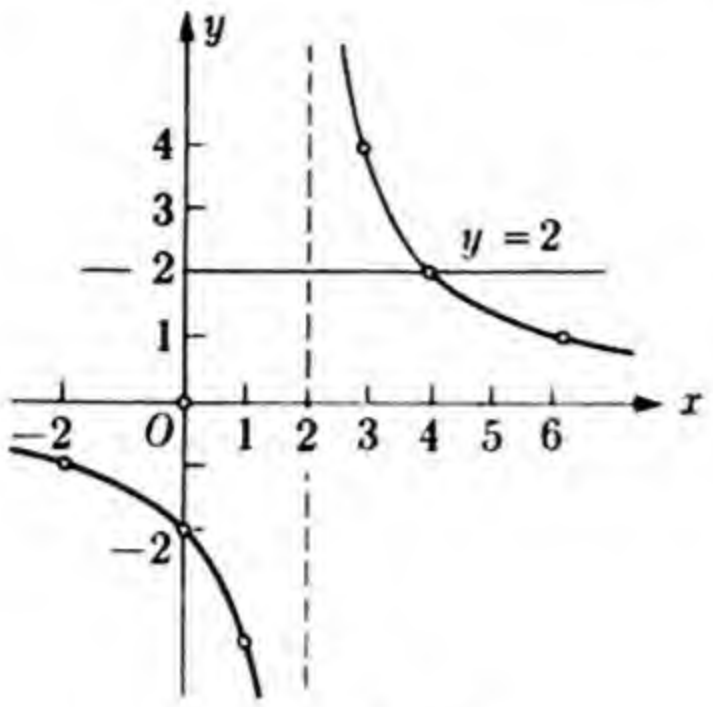


FIGURE 7–3

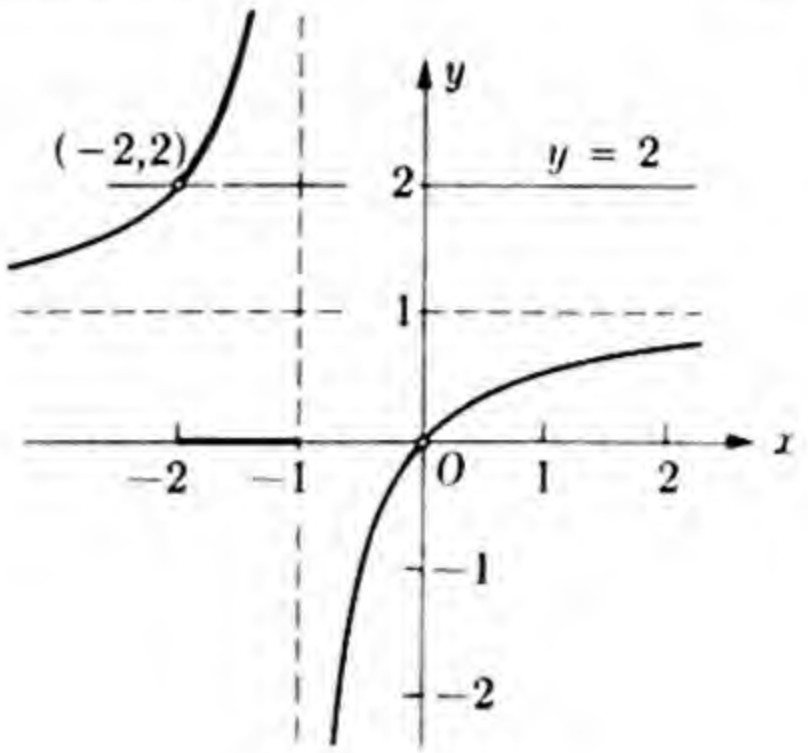


FIGURE 7–4

The inequality is not defined for $x = -1$. If $x > -1$, $x + 1$ is positive and the given inequality is equivalent to

$$x > 2x + 2 \qquad \text{or} \qquad x < -2.$$

The inequalities $x > -1$ and $x < -2$ are inconsistent.

If $x < -1$, $x + 1$ is negative and the given inequality is equivalent to

$$x < 2x + 2$$
$$x > -2.$$

Therefore, the solution of the given inequality is the set of numbers between -2 and -1, that is, the interval $-2 < x < -1$.

The equilateral hyperbola $y = x/(x + 1)$ may be sketched from its vertical asymptote $x = -1$, its horizontal asymptote $y = 1$, and the points $(0, 0)$, $(-2, 2)$. The diagram (Fig. 7–4) indicates that the hyperbola is above the line $y = 2$ for $-2 < x < -1$.

PROBLEM SET 7–4

Solve the following inequalities.

1. (a) $3x + 4 > 5$ (b) $5x - 8 > 4$
   (c) $3x - 4 < 5$ (d) $5x + 8 < -4$

2. (a) $-2x + 5 > 2$ (b) $-4x + 3 > -3$
   (c) $-3x - 5 < 4$ (d) $-4x + 10 < 6$

3. (a) $|3x - 4| > 5$ (b) $|5x + 8| < 2$
   (c) $|-2x + 5| > 2$ (d) $|-4x + 10| < 6$

Represent the answers on a number line.

4. $2 < |-2x + 3| < 4$. Represent the solutions of each inequality on a number line and combine them to obtain answers to the given problem.

5. (a) $|-2x+3| < -1$ (b) $|-2x+3| > -3$.

6. $\dfrac{4}{x-2} > 2$ (algebraically). See Fig. 7–3.

7. (a) $\dfrac{6}{x-1} > 3$ (b) $\dfrac{6}{x-1} < 2$.

Solve both algebraically and graphically.

8. (a) $\dfrac{2x}{x-2} < 1$ (b) $\dfrac{2x}{x+2} > 1$.

Solve both algebraically and graphically.

9. $-2 < \dfrac{1}{x+1} < 2$.

Solve both algebraically and graphically.

10. $\dfrac{x-2}{x+2} > 3$.

Solve both algebraically and graphically.

7–8 Quadratic inequalities. If the polynomial

$$ax^2 + bx + c, \qquad (a > 0),$$

can be written in the form

$$a\left(x + \frac{b}{2a}\right)^2 + P,$$

where P is a positive number, then the equation $ax^2 + bx + c = 0$ does not have real roots, the inequality $ax^2 + bx + c > 0$, $(a > 0)$, is satisfied for every x, and the inequality $ax^2 + bx + c < 0$ has no solution.

If the equation $ax^2 + bx + c = 0$ has real roots x_1 and x_2, and if $a > 0$, the quadratic inequality

$$ax^2 + bx + c > 0, \qquad (a > 0),$$

can be written in the form

$$(x - x_1)(x - x_2) > 0.$$

The inequality will be satisfied if both factors are positive or if both factors are negative. If $x_1 < x_2$, this happens if either $x > x_2$ or $x < x_1$.

The inequality

$$ax^2 + bx + c < 0, \qquad (a > 0)$$

becomes

$$(x - x_1)(x - x_2) < 0.$$

This inequality is satisfied if one factor is positive and the other negative or if x is between x_1 and x_2, that is, $x_1 < x < x_2$.

If a is negative, the problem is reduced to the above case by multiplication by (-1) and reversing the sign of inequality.

A special case of such problems arises in the solution of the inequality

$$|ax + b| > |cx + d|.$$

In this case, if a, b, c, d are rational numbers, the roots x_1 and x_2 are also rational and the factors can be found by trial. This inequality is equivalent to

$$\sqrt{(ax + b)^2} > \sqrt{(cx + d)^2}\,.$$

Since both of these radicals are positive (Corollary, Theorem 7–23),

$$(ax + b)^2 > (cx + d)^2$$

and the procedure above can be used.

EXAMPLE 7–4. Solve the inequality

$$|3x - 4| > |4 - x|.$$

The given inequality is replaced by

$$\sqrt{(3x - 4)^2} > \sqrt{(4 - x)^2}\,.$$

Then

$$(3x - 4)^2 > (4 - x)^2$$
$$9x^2 - 24x + 16 > 16 - 8x + x^2$$
$$8x^2 - 16x > 0$$
$$x^2 - 2x > 0$$
$$x(x - 2) > 0.$$

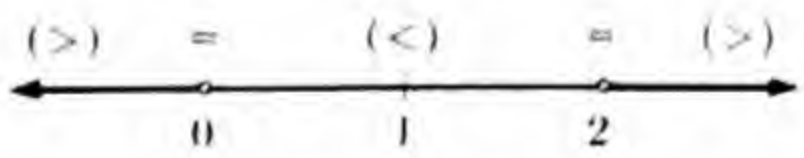


FIGURE 7–5

Hence the solution of the given problem is the set of numbers satisfying either $x > 2$ or $x < 0$. A similar analysis shows $|3x - 4| < |4 - x|$ for $0 < x < 2$. The complete situation is shown in Fig. 7–5.

EXAMPLE 7–5. Solve the inequality

$$|x + 2| > |-2x + 5|.$$

This inequality is equivalent to

$$\sqrt{(x+2)^2} > \sqrt{(-2x+5)^2}.$$

Then

$$\begin{aligned} (x+2)^2 &> (-2x+5)^2 \\ x^2+4x+4 &> 4x^2-20x+25 \\ -3x^2+24x-21 &> 0 \\ x^2-8x+7 &< 0 \\ (x-1)(x-7) &< 0. \end{aligned}$$

Hence x lies between 1 and 7 or the solution is $1 < x < 7$.

EXAMPLE 7–6. Solve the inequality

$$\frac{x}{x+1} < \frac{5}{2(x-2)}.$$

The common denominator used to clear of fractions is $2(x+1)(x-2)$. This common denominator is positive if either $x > 2$ or $x < -1$. In this case the inequality reduces to

$$\begin{aligned} 2x^2-4x &< 5x+5 \\ 2x^2-9x-5 &< 0 \\ (2x+1)(x-5) &< 0, \end{aligned}$$

which is satisfied if x is between $-\frac{1}{2}$ and 5. The conditions $x > 2$ or $x < -1$ and $-\frac{1}{2} < x < 5$ are consistent if and only if (see Fig. 7–6) $2 < x < 5$.

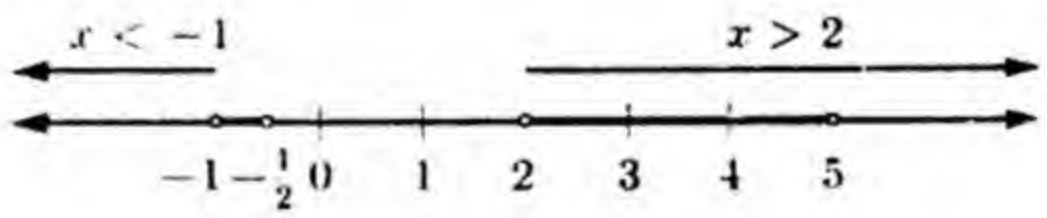


FIGURE 7-6

The common denominator $2(x+1)(x-2)$ is negative if $-1 < x < 2$. In this case, the inequality reduces to

$$\begin{aligned} 2x^2-4x &> 5x+5 \\ 2x^2-9x-5 &> 0 \\ (2x+1)(x-5) &> 0, \end{aligned}$$

which is satisfied if either $x > 5$ or $x < -\frac{1}{2}$. The two conditions on x are consistent if and only if $-1 < x < -\frac{1}{2}$.

Hence the solution of the original inequality consists of all numbers on the two intervals

$$-1 < x < -\tfrac{1}{2} \quad \text{and} \quad 2 < x < 5.$$

If the two equilateral hyperbolas

$$y_1 = \frac{x}{x + 1}, \qquad y_2 = \frac{5}{2(x - 2)}$$

are drawn in the same diagram, it will be found that $y_1 < y_2$ for these same intervals.

EXAMPLE 7–7. Solve the inequalities

$$x^2 - 6x + 4 > 0 \quad \text{and} \quad x^2 - 6x + 4 < 0.$$

The roots of the equation $x^2 - 6x + 4 = 0$ are $3 - \sqrt{5}$ and $3 + \sqrt{5}$. By trying a single point between these roots, say $x = 3$, it is found that $3^2 - 6 \cdot 3 + 4 < 0$ and hence $x^2 - 6x + 4 < 0$ for $3 - \sqrt{5} < x < 3 + \sqrt{5}$ and $x^2 - 6x + 4 > 0$ for $x > 3 + \sqrt{5}$ and for $x < 3 - \sqrt{5}$ (Fig. 7–7).

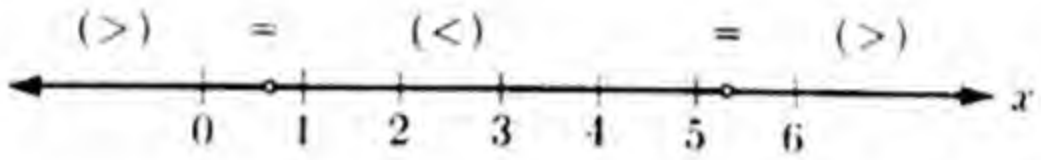


FIGURE 7–7

PROBLEM SET 7–5

1. Show by the method of completing the square that each of the following inequalities is satisfied either by every real x or by no real x.

(a) $x^2 - 4x + 8 > 0$
(b) $x^2 + 6x + 10 < 0$
(c) $2x^2 + 6x + 5 > 0$
(d) $-2x^2 + 6x - 5 > 0$
(e) $-3x^2 + 6x - 4 > 0$
(f) $-3x^2 + 6x - 4 < 0$

2. Solve the following inequalities.

(a) $(x - 2)(x + 3) > 0$
(b) $(x - 2)(x + 3) < 0$
(c) $(2x - 3)(3x + 2) > 0$
(d) $(2x - 3)(3x + 2) < 0$
(e) $(2x - 5)(5 - x) > 0$
(f) $(2x - 5)(5 - x) < 0$

3. Solve the following inequalities.

(a) $2x^2 + x - 6 > 0$
(b) $-2x^2 + 3x + 9 < 0$
(c) $x^2 - 6x + 9 > 0$
(d) $4x^2 - 12x + 9 < 0$
(e) $6x^2 + 13x - 15 < 0$
(f) $6x^2 + 13x - 15 > 0$

4. Solve the following inequalities.

(a) $x^2 + 2x - 1 > 0$
(b) $x^2 + x - 1 < 0$
(c) $2x^2 - 6x + 3 > 0$
(d) $2x^2 + x - 2 < 0$

5. Solve the following inequalities.

(a) $|x + 3| < |x - 4|$ (b) $|x + 1| > |3 - x|$

(c) $|2x - 3| < |x - 4|$ (d) $|4x - 5| > |2x - 7|$

(e) $|x + 2| < |2x - 6|$ (f) $|2x - 5| > |3x + 1|$

6. Solve the following inequalities.

(a) $\dfrac{4}{x-2} < \dfrac{8}{x+2}$. Check the results graphically.

(b) $\dfrac{2x}{x+4} > \dfrac{8}{3x+6}$

7–9 Completing the real number system. The axioms of order, and especially the fact that 1 is positive, are important in developing the real number system. The natural numbers, $1, 2 = 1 + 1, 3 = 2 + 1, 4, 5, \ldots, 10, \ldots$ are all positive and

$$0 < 1 < 2 < 3 < \cdots < 10 < \cdots.$$

If $a > b$, then $-a < -b$ (Theorem 7–22); hence

$$\cdots -10 < \cdots < -3 < -2 < -1 < 0 < 1 < 2 < 3 \cdots,$$

and it is always possible to determine which of two given integers is the greater.

The rational numbers a/b, where a and b are integers and $b \neq 0$, also obey the order axioms. Any two rational numbers can be compared as to size by writing them in equivalent forms with the same denominator and comparing numerators. This applies whether the rational number is a common fraction or a repeating decimal fraction. A decimal fraction written in the form $A.a_1a_2a_3 \ldots$, where A is a natural number, is positive, and can be compared with a similar decimal fraction $B.b_1b_2b_3 \ldots$ by recognizing that the denominator is a power of 10.

The rational numbers are *everywhere dense*. That is, between any two rational numbers there is always another rational number and, indeed, an unlimited set of rational numbers. Let r and p, $(r < p)$, be two rational numbers and let t be a rational number between 0 and 1, that is, a fraction whose numerator and denominator are natural numbers and whose numerator is less than the denominator. Then

$$x = r + t(p - r)$$

is a rational number between r and p. That the sum, difference, product and quotient of two rational numbers is a rational number follows from the laws discussed in Chapter 3. Since t and $(p - r)$ are positive, $x > r$.

The value of x can be written in the form

$$x = p + (r - p) + t(p - r)$$
$$= p - (1 - t)(p - r).$$

Since $(1 - t)$ and $(p - r)$ are positive, $x < p$, or x is between r and p: $(r < x < p)$ for an unlimited set of values of t.

The interval between two rational numbers could, for example, be divided into an integral number of equal parts by using $t = 1/m, 2/m, \ldots (m - 1)/m$, where m is a positive integer. Each such subinterval could be further subdivided to obtain more rational numbers, and there is no end to the process. These numbers could be ordered with respect to "less than."

Not all the real numbers can be obtained in this way. There are real numbers that are not rational. It was proved in Section 6-1 that $\sqrt{2}$ was not rational and suggested that the square root of many integers are not rational. It was proved in Section 3-7 that infinite decimal fractions that are not periodic are not rational. In order to complete the real number system by including such numbers, one more axiom is needed. Before stating this axiom, some preliminary definitions and discussions are given. It is beyond the scope of this text to give a complete discussion of this axiom.

DEFINITION. A non-empty set of real numbers is *bounded above* by the real number M, provided every number of the set is less than or equal to M.

The numbers of the set may be given explicitly or they may be described by a rule of formation. If the elements of the set are designated by $x_1, x_2, x_3, \ldots, x_n, \ldots$, then the set is bounded above by M if $x \leqq M$ for every x in the set. The number M is called an *upper bound* of the set. Every number greater than M is also an upper bound, and there may be upper bounds less than M.

DEFINITION. The *least upper bound* of a set of real numbers is the upper bound which is less than any other upper bound.

LEAST UPPER BOUND AXIOM. *If a non-empty set of real numbers has an upper bound, then it has a least upper bound.*

This axiom enables us to show, for example, that the set of successive approximations to $\sqrt{N}$ obtained by the division process define $\sqrt{N}$ as a real number and that every infinite decimal fraction defines a real number. A few illustrations are given here. The rational number $\frac{1}{3}$ can be expressed as the repeating decimal $0.333\ldots$. The set of numbers

0.3, 0.33, . . . , 0.333, . . . has an upper bound of 0.4. It therefore has a least upper bound which is indeed $\frac{1}{3}$.

Every decimal fraction of the form $A.a_1a_2a_3\ldots$, whether periodic or not, can be replaced by the set

$$A.a_1, \qquad A.a_1a_2, \qquad A_1.a_1a_2a_3, \qquad \ldots$$

which has the upper bound $A + 1$ and which therefore has a least upper bound that can be approximated to any required accuracy by using enough decimal places. This least upper bound is a real number defined by the decimal fraction.

From the point of view of infinite decimal fractions, it is not difficult to see that between any two real numbers (rational or irrational), there are infinitely many rational and irrational numbers. It was for this reason that Section 7–1 suggested the set of all infinite decimal fractions as a model of the real number system.

In summary, the real numbers are entities which obey: (1) the axioms of a number field, that is, the Axioms of Equality E1–E4, the Axioms of Addition A1–A5, and the Axioms of Multiplication M1–M6; (2) the Axioms of Order O1–O3; (3) the Least Upper Bound Axiom.

There are other number systems. The system of complex numbers was mentioned briefly in Section 6–5. Complex numbers do obey the axioms of a number field but do not obey the axioms of order. In the next chapter an extension of the number system is made. The new type of number does not obey all the axioms of a number field, and it is important to determine which axioms these numbers do or do not obey.

CHAPTER 8

INTRODUCTION TO MATRIX ALGEBRA AND LINEAR PROGRAMMING

8–1 Row vectors and column vectors. In this chapter, new types of numbers are introduced by means of definitions. The relation of equality and the operations of addition and multiplication for these new numbers, represented by the usual symbols, $=$, $+$, $\cdot$, or by placing the numbers in juxtaposition, are also introduced through definitions. It will be determined which of the laws for real numbers remain valid and which of these laws are no longer valid. It will be shown that all of the laws of addition are valid under the appropriate definitions, and that some but *not all* of the laws of multiplication are valid.

DEFINITION 8–1. A *row vector* is an ordered set of numbers written in a row and enclosed in parentheses. The individual numbers of the vector are called its *components*.

In this book these components are real numbers, and hereafter this is to be understood. In writing the vector, a space is left between any two components. Examples of row vectors are $(1\ 3)$, $(1\ 3\ -2)$; and if the row vector has n components, it can be represented by $\mathbf{U} = (u_1\ u_2\ \ldots\ u_n)$, where $\mathbf{U}$ is a single letter representing the vector and $u_1, u_2, \ldots, u_n$ are real numbers. The notation $A(x, y)$ or $A(x, y, z)$, written with commas, is reserved to represent a point and its coordinates in two- or three-dimensional space. Thus $A(1, 3, -2)$ indicates a point in three-space whose x, y, and z coordinates are 1, 3, and -2, respectively.

DEFINITION 8–2. A *column vector* is an ordered set of numbers, called *components*, written in a column and enclosed in parentheses.

Examples of column vectors are

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

and, more generally,

$$\mathbf{V} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

where $\mathbf{V}$ is a single letter representing the column vector and $v_1, v_2, \ldots, v_n$

are real numbers. To avoid the inconvenience of writing long columns, a column vector is sometimes written like a row vector with the exponent T (for "transposed"). Thus

$$\begin{pmatrix}1\\3\end{pmatrix} = (1\ 3)^T, \qquad \begin{pmatrix}1\\3\\-2\end{pmatrix} = (1\ 3\ -2)^T,$$

and the general column vector $\mathbf{V} = (v_1\ v_2\ \ldots\ v_n)^T$.

DEFINITION 8–3. Two row vectors or two column vectors are *equal* if they have the same number of components and if corresponding components of the vectors are equal.

Since equality of vectors depends only upon the properties of equality for real numbers, it follows that the laws of equality E1 to E4 are also true for row and column vectors as well as for real numbers.

The definition does not include two row vectors having different numbers of components. Neither does it apply to one row vector and one column vector. For example, if

$$\mathbf{U} = (1\ 3), \qquad \mathbf{V} = (1\ 3\ 0), \qquad \mathbf{W} = (4{-}3\ 9/3), \qquad \mathbf{S} = \begin{pmatrix}1\\3\end{pmatrix},$$

then

$$\mathbf{U} = \mathbf{W}, \qquad \mathbf{U} \neq \mathbf{V}, \qquad \mathbf{U} \neq \mathbf{S}.$$

DEFINITION 8–4. If $\mathbf{U}$ and $\mathbf{V}$ are two vectors of the same type, their *sum*, indicated by $\mathbf{U} + \mathbf{V}$, is a vector of the same type whose components are the sum of the corresponding components of $\mathbf{U}$ and $\mathbf{V}$.

By "the same type" (or "conformable") is meant that both are row vectors or both are column vectors and that the vectors have the same number of components. Otherwise, addition has not been defined.

EXAMPLE 8–1.

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} + \begin{pmatrix}2\\-1\\4\end{pmatrix} = \begin{pmatrix}3\\2\\2\end{pmatrix}$$

and

$$(1\ 3\ -2) + (2\ -1\ 4) = (3\ 2\ 2).$$

$(1\ 3\ -2) + (2\ -1)$ is meaningless.

In general,

$$(u_1\ u_2\ \ldots\ u_n) + (v_1\ v_2\ \ldots\ v_n) = (u_1 + v_1\ u_2 + v_2\ \ldots\ u_n + v_n),$$

with a similar law for column vectors.

Since addition of real numbers is closed, commutative, and associative, it follows that these same properties hold for conformable vectors. (Inasmuch as addition is not defined except for conformable vectors, the adjective is omitted except when used for emphasis.) If **U**, **V**, **W** are conformable vectors, then

$$\mathbf{U} + \mathbf{V} \text{ is a vector of the same type,} \tag{8-1}$$

$$\mathbf{U} + \mathbf{V} = \mathbf{V} + \mathbf{U}, \tag{8-2}$$

$$(\mathbf{U} + \mathbf{V}) + \mathbf{W} = \mathbf{U} + (\mathbf{V} + \mathbf{W}) = \mathbf{U} + \mathbf{V} + \mathbf{W}. \tag{8-3}$$

DEFINITION 8-5. A *zero vector* is one all of whose components are zero.

When any zero vector is combined with other vectors, it is understood to represent a vector conformable with them. When necessary, the number of components can be indicated by a subscript. Thus $\mathbf{0}_3$ would be used for either

$$(0\ 0\ 0) \qquad \text{or} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The zero vector has the important property

$$\mathbf{U} + \mathbf{0} = \mathbf{U}. \tag{8-4}$$

This is an immediate consequence of the properties of real numbers: $u + 0 = u$.

DEFINITION 8-6. The *additive inverse*, or the *negative*, of a vector **U** is the vector whose components are the negatives of the components of **U**, and this is indicated by $(-\mathbf{U})$.

If $\mathbf{U} = (u_1\ u_2\ u_3)$, then $(-\mathbf{U}) = (-u_1\ -u_2\ -u_3)$, with similar statements for column vectors or for vectors with n components. It is easy to verify the property

$$\mathbf{U} + (-\mathbf{U}) = \mathbf{0}. \tag{8-5}$$

Since vectors do not obey the Axioms of Order given for real numbers, positive and negative vectors are not defined. As indicated in Definition 8-6, the *negative of a vector* is introduced.

The operation of *subtraction* of conformable vectors, indicated by the use of the minus sign, $-$, is defined as follows:

DEFINITION 8-7. If

$$\mathbf{X} + \mathbf{U} = \mathbf{V}, \qquad \text{then} \qquad \mathbf{X} = \mathbf{V} - \mathbf{U}. \tag{8-6}$$

If the components of these vectors are x_i, u_i, v_i, $(i = 1, 2, \ldots, n)$, respectively, then

$$x_i + u_i = v_i \qquad \text{and} \qquad x_i = v_i - u_i.$$

Since $v_i - u_i = v_i + (-u_i)$, where $(-u_i)$ is the additive inverse of u_i, it follows that also

$$\mathbf{V} - \mathbf{U} = \mathbf{V} + (-\mathbf{U}), \tag{8-7}$$

where $(-\mathbf{U})$ is the additive inverse of $\mathbf{U}$.

THEOREM 8-1.* *If* $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ *are conformable vectors, and if* $\mathbf{U} = \mathbf{V}$, *then* $\mathbf{U} + \mathbf{W} = \mathbf{V} + \mathbf{W}$ *and, conversely, if* $\mathbf{U} + \mathbf{W} = \mathbf{V} + \mathbf{W}$, *then* $\mathbf{U} = \mathbf{V}$. Thus all the fundamental laws of addition are valid for real vectors as well as for real numbers.

DEFINITION 8-8. The *product of a number and a vector* is a vector whose components are the components of the given vector each multiplied by the given number:

$$k\mathbf{U} = \mathbf{U}k = k(u_1\ u_2\ \ldots\ u_n) = (ku_1\ ku_2\ \ldots\ ku_n),$$

with similar formulas for a column vector, where k is a given real number.

Such multiplication is referred to as *scalar multiplication* or *numerical multiplication* to distinguish it from other types of multiplication which involve vectors.

As a consequence of this definition, it is easy to see that

$$0\mathbf{U} = \mathbf{0}; \qquad 1\mathbf{U} = \mathbf{U}; \qquad (-1)\mathbf{U} = (-\mathbf{U}) \tag{8-8}$$

and that

$$\mathbf{V} - \mathbf{U} = \mathbf{V} + (-1)\mathbf{U}. \tag{8-9}$$

The minus sign, $-$, has the same three interpretations with vectors that it has with real numbers.

THEOREM 8-2.* *If* a *and* b *are numbers and* $\mathbf{U}$ *and* $\mathbf{V}$ *are vectors, then*

$$(a + b)\mathbf{U} = a\mathbf{U} + b\mathbf{U}, \tag{8-10}$$

$$a(b\mathbf{U}) = (ab)\mathbf{U} = b(a\mathbf{U}), \tag{8-11}$$

$$a(\mathbf{U} + \mathbf{V}) = a\mathbf{U} + a\mathbf{V}. \tag{8-12}$$

---

* The proof is left as an exercise.

Problem Set 8-1

1. If $\mathbf{U} = (2\ 3)$, $\mathbf{V} = (-4\ 2)$, $\mathbf{W} = (-1\ 0)$, compute (a) $\mathbf{U} + \mathbf{V}$; (b) $\mathbf{U} - \mathbf{V}$; (c) $2\mathbf{U} + 3\mathbf{W}$; (d) $\mathbf{X}$, if $3\mathbf{V} + \mathbf{X} = 2\mathbf{W}$.

2. If $\mathbf{U} = (2\ 3\ 1)$, $\mathbf{V} = (-4\ -2\ 0)$, $\mathbf{W} = (1\ 0\ -1)$, compute (a) $\mathbf{V} + \mathbf{U} - \mathbf{W}$; (b) $2\mathbf{W} - 3\mathbf{V} + 4\mathbf{U}$; (c) $\mathbf{Y}$, if $3\mathbf{V} + 2\mathbf{W} + \mathbf{Y} = 0$.

3. If

$$\mathbf{U} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix},$$

compute (a) $\mathbf{U} + \mathbf{V}$; (b) $\mathbf{V} - \mathbf{U}$; (c) $\frac{1}{2}\mathbf{U} + \frac{1}{4}\mathbf{V}$; (d) $\mathbf{X}$, if $2\mathbf{V} + 3\mathbf{X} = \mathbf{U}$.

4. Explain why it is not possible to compute the following sums.

(a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ (b) $(-1\ 1) + 1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

5. Prove the Associative Law of Addition for vectors:

$$(\mathbf{U} + \mathbf{V}) + \mathbf{W} = \mathbf{U} + (\mathbf{V} + \mathbf{W}).$$

6. Prove the Cancellation Law of Addition for vectors and its converse (Theorem 8-1).

7. Prove (a) $k\mathbf{0} = \mathbf{0}$; (b) $0\mathbf{U} = \mathbf{0}$ for any $\mathbf{U}$; (c) $\mathbf{U} + 0\mathbf{V} = \mathbf{U}$.

8. Prove the equations

$$(-1)\mathbf{U} = (-\mathbf{U}),$$
$$\mathbf{V} - \mathbf{U} = \mathbf{V} + (-1)\mathbf{U}.$$

9. Prove the equations of Theorem 8-2:

$$(a + b)\mathbf{U} = a\mathbf{U} + b\mathbf{U},$$
$$a(b\mathbf{U}) = (ab)\mathbf{U} = b(a\mathbf{U}),$$
$$a(\mathbf{U} + \mathbf{V}) = a\mathbf{U} + b\mathbf{V}.$$

State these laws in words.

10. If the quantities of certain commodities involved in a budget equation are x_1, x_2, x_3, would it be reasonable to represent this set of quantities by a vector? If the corresponding prices are p_1, p_2, p_3, would it be reasonable to represent this set of prices by a vector of the same or different type? Would it be reasonable to add two such vectors?

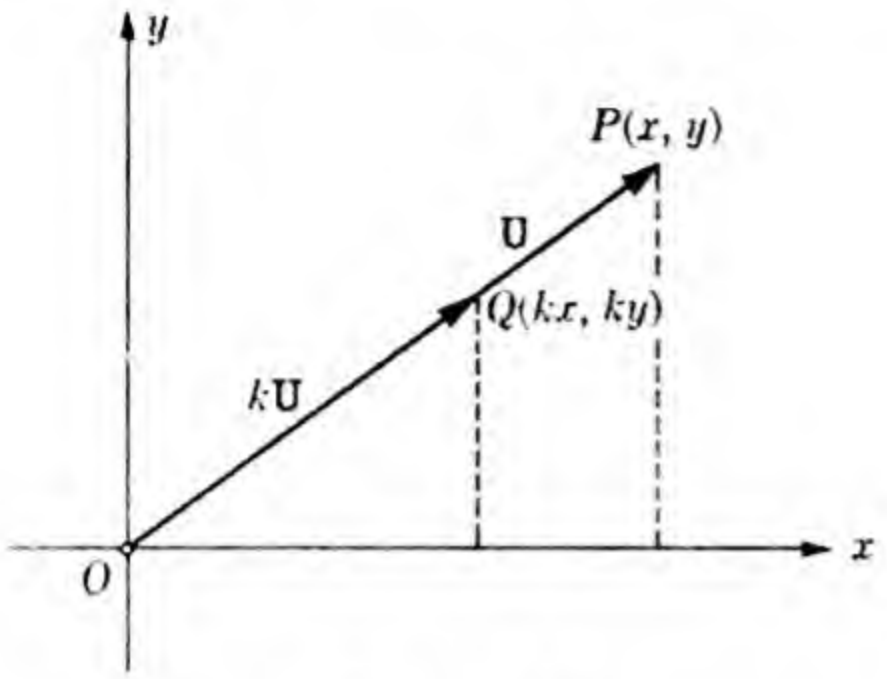


FIGURE 8–1

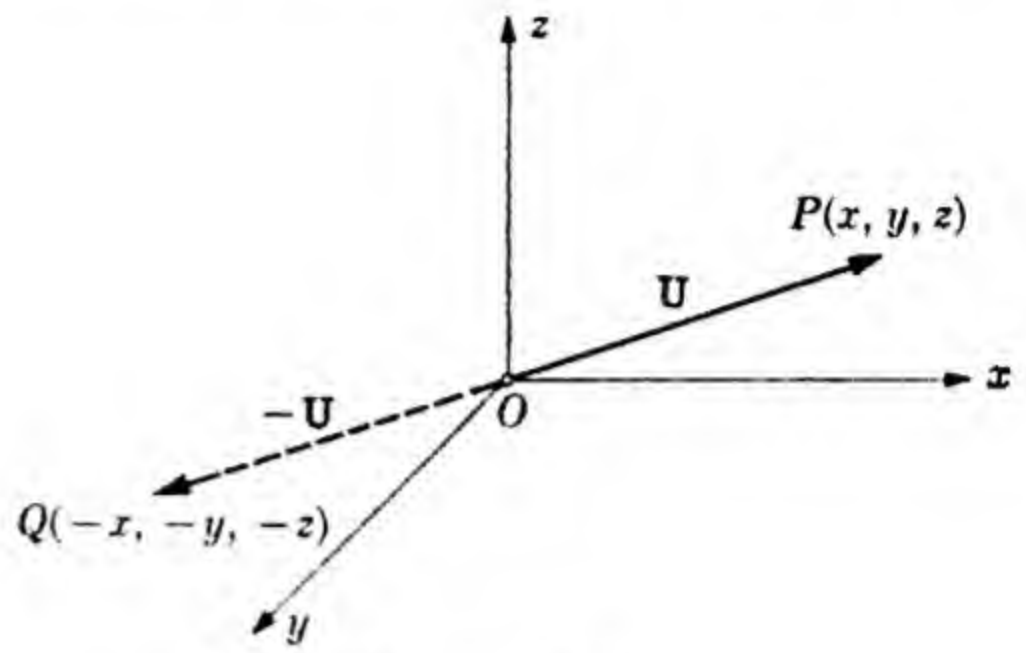


FIGURE 8–2

8–2 Geometric interpretation of vectors. With either a row or column vector **U** which has two components, there is associated a point P which has these components as coordinates (Fig. 8–1). The directed line segment $\overrightarrow{OP}$ is used to represent the vector **U**. Similarly (Fig. 8–2), if **U** (or its transpose $\mathbf{U}^T$) is the vector $(x\ y\ z)$, the directed line segment $\overrightarrow{OP}$ from the origin $O(0, 0, 0)$ to $P(x, y, z)$ is used to represent the vector **U** (or its transpose). For every vector except the zero vector, there corresponds in a one-to-one way a directed line segment. The zero vector is represented by the origin and there is no direction that corresponds to it. Such a representation is called a *geometric vector* (*bound to the origin*)† when it is desired to distinguish between the algebraic and geometric representation of the vector.

The vector $k\mathbf{U}$ is a vector whose terminal point has coordinates $Q(kx, ky, kz)$ and hence which lies on the line OP; the directions of $k\mathbf{U}$ and **U** are the same if $k > 0$ and opposite if k is negative. Figure 8–1 shows a situation where $k = \frac{2}{3}$, and Fig. 8–2 shows the case when $k = -1$. Conversely, two geometric vectors are collinear only if one is a numerical multiple of the other. These statements are immediate consequences of the theory of similar triangles (Fig. 8–1).

THEOREM 8–3. PARALLELOGRAM LAW. *If* **U** *and* **V** *are noncollinear geometric vectors, then the sum* $\mathbf{U} + \mathbf{V}$ *is the vector which is the diagonal of the parallelogram which has* **U** *and* **V** *as sides.*

The proof for the two-dimensional case is given here; that for the three-dimensional case is quite similar, but involves several sets of congruent triangles. Let the terminals of vectors **U**, **V** be $A(x_1, y_1)$, $B(x_2, y_2)$, respectively. Let $C(x_3, y_3)$ be the diagonal of the parallelogram $OACB$. Let BM (Fig. 8–3) and CN be parallel to the y-axis and AN parallel to

† The qualifying phrase *bound to the origin* is hereafter omitted because this is the only geometric representation considered in this book.

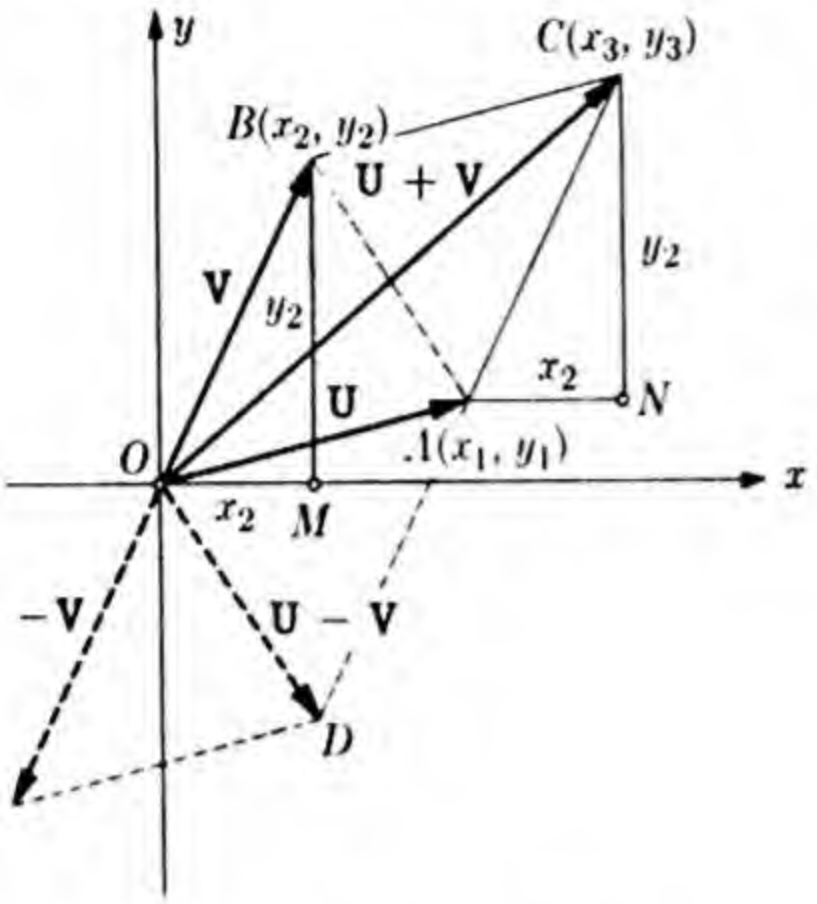


FIGURE 8–3

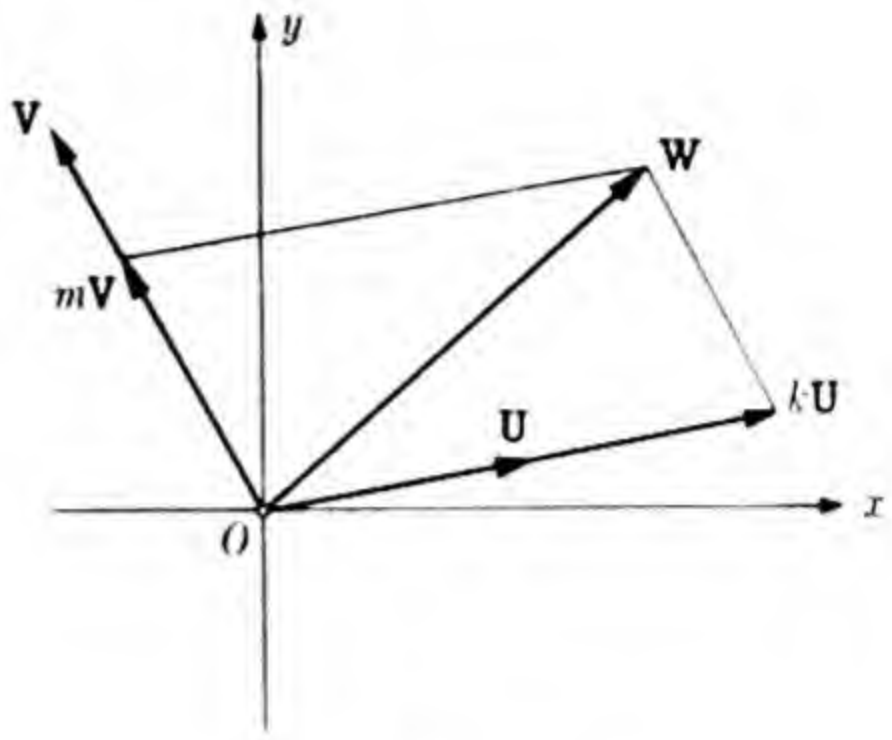


FIGURE 8–4

the x-axis. The right triangles OMB and ANC are congruent, since $OB \cong AC$ and $\angle OBM \cong \angle ACN$. It follows that $AN = x_2$, $NC = y_2$, and

$$x_3 = x_1 + x_2, \qquad y_3 = y_1 + y_2,$$

so that the geometric vector $\overrightarrow{OC}$ is $\mathbf{U} + \mathbf{V}$.

The vector $-\mathbf{V}$ is directed opposite to $\mathbf{V}$, and a similar construction shows that $\mathbf{U} - \mathbf{V}$ is the diagonal $\overrightarrow{OD}$ of the parallelogram that has $\mathbf{U}$ and $-\mathbf{V}$ as sides. It is not difficult to prove that the line segment $\overrightarrow{OD}$ is parallel and equal to $\overrightarrow{BA}$, and this idea could be used to construct $\mathbf{U} - \mathbf{V}$.

The parallelogram law makes it possible to construct any linear combination of $\mathbf{U}$ and $\mathbf{V}$, namely, $\mathbf{W} = k\mathbf{U} + m\mathbf{V}$. The vector $k\mathbf{U}$ is a multiple of $\mathbf{U}$, $m\mathbf{V}$ is a multiple of $\mathbf{V}$ and their sum is then readily constructed. Conversely, if $\mathbf{U}$ and $\mathbf{V}$ are noncollinear vectors, it is possible to determine the numbers k and m so that $\mathbf{W} = k\mathbf{U} + m\mathbf{V}$, where $\mathbf{W}$ is any vector in the plane determined by the geometric vectors $\mathbf{U}$ and $\mathbf{V}$. Through the terminal of $\mathbf{W}$ draw parallels to $\overrightarrow{OU}$ and $\overrightarrow{OV}$, thus determining vectors $\overrightarrow{OK} = k\mathbf{U}$ and $\overrightarrow{OM} = m\mathbf{W}$ (Fig. 8–4) so that $\mathbf{W} = k\mathbf{U} + m\mathbf{V}$. If the given vectors are $\mathbf{U} = (x_1\ y_1)$, $\mathbf{V} = (x_2\ y_2)$, $\mathbf{W} = (x_3\ y_3)$, the algebraic solution of the problem is obtained by solving the simultaneous equations

$$\begin{aligned} kx_1 + mx_2 &= x_3 \\ ky_1 + my_2 &= y_3. \end{aligned}$$

Since the vectors $\mathbf{U}$ and $\mathbf{V}$ are noncollinear, the determinant

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \neq 0$$

and the solution can always be found by Cramer's rule or other convenient methods.

EXAMPLE 8–2. Express the vector (5 5) as a linear combination of the vectors (3 1) and (−1 4). (See Fig. 8–4.)

The equations to be solved are

$$3k - m = 5,$$
$$k + 4m = 5.$$

Hence

$$k = \frac{\begin{vmatrix} 5 & -1 \\ 5 & 4 \end{vmatrix}}{13} = \frac{25}{13} \qquad m = \frac{\begin{vmatrix} 3 & 5 \\ 1 & 5 \end{vmatrix}}{13} = \frac{10}{13},$$

so that

$$(5\ 5) = \frac{25}{13}(3\ 1) + \frac{10}{13}(-1\ 4).$$

The following theorems in 3-space, corresponding to the parallelogram law and its consequences, are stated without proof.

THEOREM 8–4. *If* **U, V, W** *are three geometric vectors which do not lie in the same plane, then:* (1) *the sum* **U** + **V** + **W** *is the vector which is the diagonal of the parallelepiped which has* **U**, **V**, **W** *as edges,* (2) *any vector* **Z** *can be expressed as a linear combination of* **U**, **V**, *and* **W**.

EXAMPLE 8–3. If **U** = (1 0 −1), **V** = (1 1 1), **W** = (0 2 −2), and **Z** = (1 4 −3), express **Z** in the form $\mathbf{Z} = k\mathbf{U} + m\mathbf{V} + p\mathbf{W}$.

This vector equation is equivalent to the three linear equations

$$\begin{aligned} k + m &= 1 \\ m + 2p &= 4 \\ -k + m - 2p &= -3. \end{aligned}$$

The solution in terms of third-order determinants (Section 5–8) is $k = D_1/D$, $m = D_2/D$, $p = D_3/D$, where

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{vmatrix} = -6 \neq 0,$$

$$D_1 = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 1 & 2 \\ -3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ -3 & 1 & -2 \end{vmatrix} = -2, \quad \text{so that } k = \tfrac{1}{3}.$$

D_2 and D_3 could also be computed, but the first equation yields $m = \frac{2}{3}$,

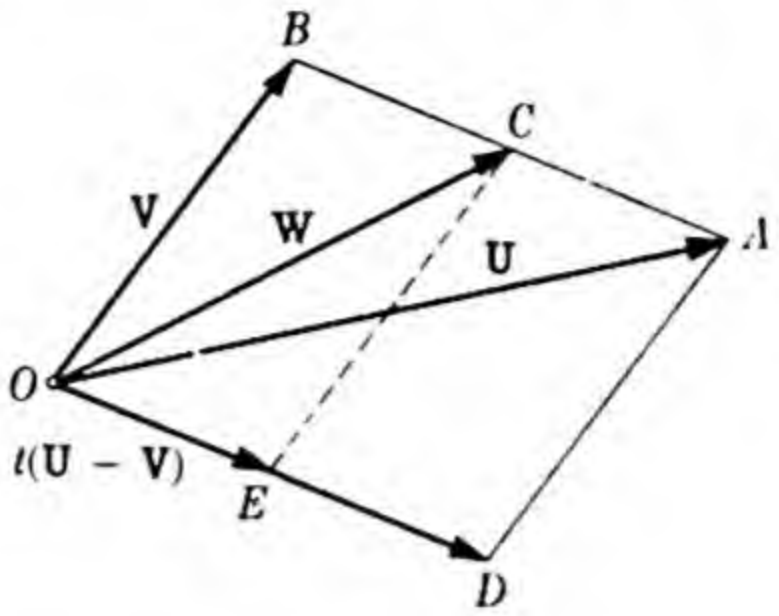


FIGURE 8-5

and then the second equation gives $p = \frac{1}{2}(4 - \frac{2}{3}) = \frac{5}{3}$. The last equation is used as a check:

$$-\tfrac{1}{3} + \tfrac{2}{3} - \tfrac{10}{3} = -\tfrac{9}{3} = -3.$$

The following theorem and its proof are valid in either 2- or 3-dimensional space. One of the advantages of the vector notation is that it is possible to denote a set of numbers by a single letter and it is often immaterial how many numbers are in the set. Indeed, the following theorem is valid in n-dimensional space, where the word "between" is given an algebraic rather than a geometric interpretation.

THEOREM 8-5. *If* $\mathbf{U} = \overrightarrow{OA}$ *and* $\mathbf{V} = \overrightarrow{OB}$ *are geometric vectors that are not collinear, and if* C *is any point between* A *and* B, *then*

$$\mathbf{W} = \overrightarrow{OC} = t\mathbf{U} + (1 - t)\mathbf{V}, \qquad (0 < t < 1).$$

Conversely, if this equation is satisfied by $\mathbf{W}$, *then* C *is between* A *and* B.

Proof. Let $\overrightarrow{OD}$ be the vector $\mathbf{U} - \mathbf{V}$, so that $ODAB$ is a parallelogram (see remark after Theorem 8-3). Through the point C draw a parallel to the line AD, determining the point E and the vector $\overrightarrow{OE} = t(\mathbf{U} - \mathbf{V})$, where $0 < t < 1$. E is between O and D because C is between A and B. From the parallelogram $OECB$ (Fig. 8-5), it follows that

$$\mathbf{W} = t(\mathbf{U} - \mathbf{V}) + \mathbf{V}$$

or

$$\mathbf{W} = t\mathbf{U} + (1 - t)\mathbf{V}, \qquad (0 < t < 1). \tag{8-13}$$

Conversely, if $\mathbf{W} = \overrightarrow{OC}$ is a vector that satisfies Eq. (8-13) or $\mathbf{W} = t(\mathbf{U} - \mathbf{V}) + \mathbf{V}$, then $\overrightarrow{OC}$ is the diagonal of the parallelogram that has $\overrightarrow{OE}$ and $\overrightarrow{OB}$ as its sides. C lies on the line AB between A and B, since E is between O and D.

Note that $t = 0$ in Eq. (8-13) yields $\mathbf{W} = \mathbf{V}$ and $t = 1$ yields $\mathbf{W} = \mathbf{U}$.

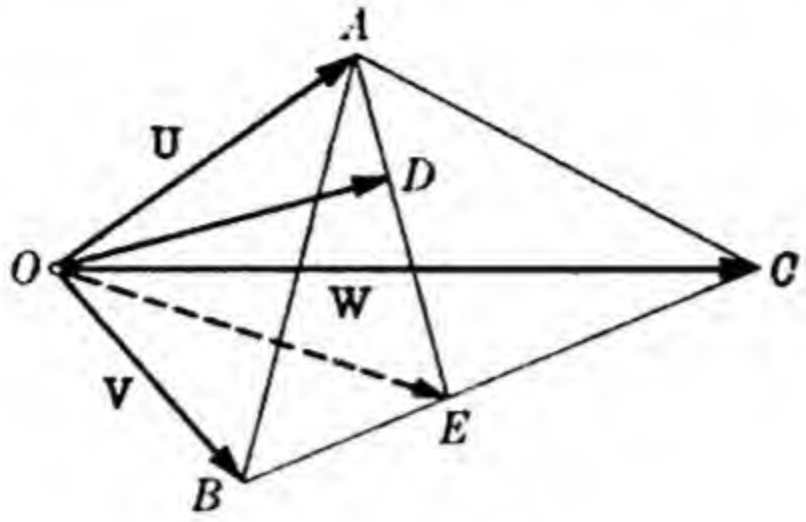


FIGURE 8-6

COROLLARY 1. *If Q is the midpoint of the segment $\overline{AB}$, then* $\mathbf{W} = \overrightarrow{OQ} = (\mathbf{U} + \mathbf{V})/2$.

COROLLARY 2. *If* $\mathbf{U} = (x_1\ y_1)$ *and* $\mathbf{V} = (x_2\ y_2)$, *Eq.* (8-13) *is equivalent to the two equations*

$$\begin{aligned} x_3 &= tx_1 + (1 - t)x_2, \\ y_3 &= ty_1 + (1 - t)y_2, \qquad (0 < t < 1), \end{aligned} \tag{8-14}$$

and if $\mathbf{U} = (x_1\ y_1\ z_1)$ *and* $\mathbf{V} = (x_2\ y_2\ z_2)$, *Eq.* (8-13) *is equivalent to the three equations*

$$\begin{aligned} x_3 &= tx_1 + (1 - t)x_2, \\ y_3 &= ty_1 + (1 - t)y_2, \\ z_3 &= tz_1 + (1 - t)z_3, \qquad (0 < t < 1). \end{aligned} \tag{8-15}$$

Theorems 8-5 and 8-6 play an important role in the theory of linear inequalities and linear programming (Sections 8-8 and 8-9).

THEOREM 8-6. *If* $\mathbf{U} = \overrightarrow{OA}$, $\mathbf{V} = \overrightarrow{OB}$, $\mathbf{W} = \overrightarrow{OC}$ *are geometric vectors such that A, B, C form a triangle, and if D is a point in the interior of the triangle, then*

$$\mathbf{Z} = \overrightarrow{OD} = t\mathbf{U} + s\mathbf{V} + (1 - t - s)\mathbf{W}, \qquad (0 < t < 1,\ 0 < s < 1,\ 0 < 1 - t - s < 1). \tag{8-16}$$

Conversely, if $\mathbf{Z} = \overrightarrow{OD}$ *satisfies Eq.* (8-16), *then D is inside the triangle ABC.*

Proof. The point D is inside the triangle ABC if and only if it is between the vertex A and a point E which is between B and C. The point E is between B and C if and only if $\overrightarrow{OE} = r\mathbf{V} + (1 - r)\mathbf{W}$, $(0 < r < 1)$; and the point D is in the interior of the triangle if and only if

$$\mathbf{Z} = \overrightarrow{OD} = t\mathbf{U} + (1 - t)r\mathbf{V} + (1 - t)(1 - r)\mathbf{W}, \qquad (0 < t < 1).$$

Let $(1 - t)r = s$. Since $(1 - t)$ and r are both between 0 and 1, s is also. Further, $0 < (1 - t)(1 - r) < 1$ and

$$(1 - t)(1 - r) = 1 - t - (1 - t)r = 1 - t - s.$$

Therefore

$$\mathbf{Z} = \overrightarrow{OD} = t\mathbf{U} + s\mathbf{V} + (1 - t - s)\mathbf{W},$$

$(0 < t < 1,\ 0 < s < 1,\ 0 < 1 - s - t < 1)$, if and only if D is in the interior of triangle ABC.

The proof is valid if A, B, C are in the xy-plane or in 3-space. The equations in terms of coordinates that correspond to the vector Eq. (8–16), in both two and three dimensions, are readily given.

Problem Set 8–2

1. If $\mathbf{U} = (2\ 3)$, $\mathbf{V} = (-4\ 2)$ and $\mathbf{W} = (-1\ 0)$, determine the following vectors geometrically.

(a) $\mathbf{U} + \mathbf{V}$, (b) $\mathbf{U} - \mathbf{V}$, (c) $2\mathbf{U} + 3\mathbf{W}$, (d) $2\mathbf{W} - 3\mathbf{U}$.

2. If

$$\mathbf{U} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix},$$

in a perspective diagram sketch the following, based upon geometric constructions:

(a) $\mathbf{U} + \mathbf{V}$, (b) $\mathbf{V} - \mathbf{U}$, (c) $\frac{1}{2}\mathbf{U} + \frac{1}{4}\mathbf{V}$, (d) $\frac{1}{3}(\mathbf{U} - 2\mathbf{V})$.

3. Express each of the following vectors as a linear combination of $\mathbf{U} = (-3\ 2)$ and $\mathbf{V} = (1\ 1)$. Solve algebraically and check graphically.

(a) $(-2\ 6)$ (b) $(6\ 1)$ (c) $(2\ -4)$

4. If $\mathbf{U} = (1\ -3)$, $\mathbf{V} = (4\ -1)$, $\mathbf{X} = (18\ 1)$, $\mathbf{Y} = (-5\ -7)$, express $\mathbf{X}$ and $\mathbf{Y}$ as linear combinations of $\mathbf{U}$ and $\mathbf{V}$ and then solve these vector equations to find $\mathbf{U}$ and $\mathbf{V}$ as linear combinations of $\mathbf{X}$ and $\mathbf{Y}$.

5. If $\mathbf{U} = (1\ 0\ -2)$, $\mathbf{V} = (1\ 2\ -1)$, $\mathbf{W} = (0\ 1\ 2)$, express each of the following vectors as a linear combination of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$. Solve algebraically only.

(a) $(4\ 2\ 1)$ (b) $(-3\ -1\ 4)$

6. (a) Show that it is possible to express the vectors $\mathbf{X} = (10\ -6\ 10)$, $\mathbf{Y} = (-9\ 7\ 7)$ as a linear combination of $\mathbf{U} = (2\ -1\ 4)$ and $\mathbf{V} = (-3\ 2\ -1)$. What geometric relation holds for these four vectors? (b) Solve these vector equations to find $\mathbf{U}$ and $\mathbf{V}$ as linear combinations of $\mathbf{X}$ and $\mathbf{Y}$.

7. If $\mathbf{U} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ and $\mathbf{V} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$,

determine the vector $\mathbf{W} = t\mathbf{U} + (1 - t)\mathbf{V}$ for the following values of t and illustrate geometrically.

(a) $t = \frac{1}{3}$ (b) $t = \frac{1}{2}$ (c) $t = \frac{3}{4}$ (d) $t = 1$
(e) $t = 0$ (f) $t = 2$ (g) $t = -2$

8. Prove Corollary 1 of Theorem 8-5: If Q is the midpoint of the segment $\overline{AB}$, where $\mathbf{U} = \overrightarrow{OA}$ and $\mathbf{V} = \overrightarrow{OB}$, then $\mathbf{W} = \overrightarrow{OQ} = \frac{1}{2}(\mathbf{U} + \mathbf{V})$.

9. (a) What can you say about the vector $\mathbf{W} = 2\mathbf{U} - \mathbf{V}$ in relation to the vector $\mathbf{U} = \overrightarrow{OA}$ and $\mathbf{V} = \overrightarrow{OB}$? About the vector $\mathbf{Z} = -\mathbf{U} + 2\mathbf{V}$? (b) Discuss the general situation for the vector $t\mathbf{U} + (1 - t)\mathbf{V}$ for $t < 0$ and for $t > 1$.

10. If $\mathbf{U} = (2 \; -1 \; 4)$ and $\mathbf{V} = (-3 \; 2 \; -1)$, determine the vector

$$\mathbf{W} = t\mathbf{U} + (1 - t)\mathbf{V}$$

for the following values of t and discuss the relative positions of the terminal points of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$.

(a) $t = \frac{3}{3}$ (b) $t = -\frac{2}{3}$ (c) $t = \frac{4}{3}$

11. (a) If $\mathbf{U} = \overrightarrow{OA}$, $\mathbf{V} = \overrightarrow{OB}$, $\mathbf{W} = \overrightarrow{OC}$, $\mathbf{Z} = \overrightarrow{OD}$ are geometric vectors in the xy-plane, write the equations in terms of coordinates which correspond to

$$\mathbf{Z} = t\mathbf{U} + s\mathbf{V} + (1 - t - s)\mathbf{W}.$$

(b) D is in the interior of triangle ABC if and only if $0 < t < 1$, $0 < s < 1$, $0 < (1 - t - s) < 1$. Using the equations in (a), show that the point $(\frac{10}{3}, \frac{8}{3})$ is in the interior of triangle ABC, where $A(2, 2)$, $B(6, 1)$, $C(2, 4)$.

(c) Using the equations in (a) and the points of (b), show that the point $(\frac{14}{3}, \frac{8}{3})$ is not in the interior of triangle ABC.

12. (a) If $\mathbf{U} = \overrightarrow{OA}$, $\mathbf{V} = \overrightarrow{OB}$, $\mathbf{W} = \overrightarrow{OC}$, $\mathbf{Z} = \overrightarrow{OD}$ are geometric vectors in 3-space, write the equations in terms of coordinates which correspond to

$$\mathbf{Z} = t\mathbf{U} + s\mathbf{V} + r\mathbf{W},$$
$$(0 < t < 1, \quad 0 < s < 1, \quad 0 < r < 1, \quad r + s + t = 1).$$

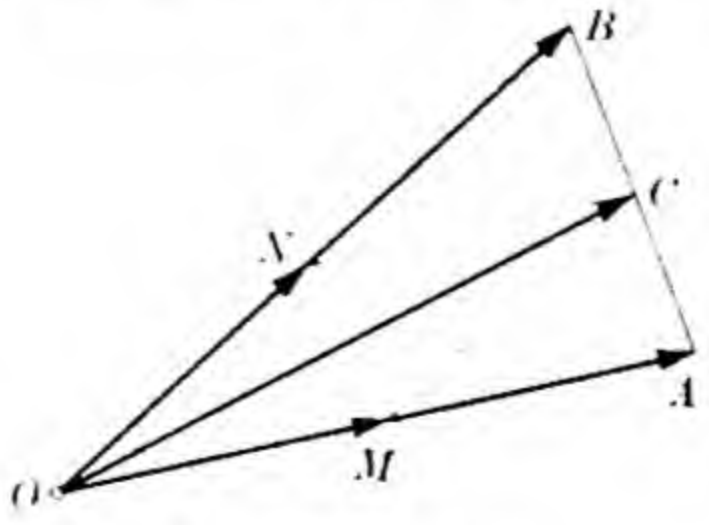


FIGURE 8-7

(b) Using these equations, show that $D(\frac{11}{6}, \frac{5}{6}, \frac{1}{3})$ is in the interior of triangle ABC, where the points are $A(1, 1, 1)$, $B(2, 0, 0)$, $C(3, 3, 0)$.

13. Prove the first part of Theorem 8-5, using Fig. 8-7: If $\mathbf{U} = \overrightarrow{OA}$ and $\mathbf{V} = \overrightarrow{OB}$ are geometric vectors that are not collinear, and if C is any point between A and B, then $\mathbf{W} = \overrightarrow{OC} = t\mathbf{U} + (1 - t)\mathbf{V}$, $(0 < t < 1)$.

Draw lines through C parallel to $\overline{OB}$ and $\overline{OA}$ and let $OM = t\mathbf{U}$. Use the theory of similar triangles.

8-3 Matrices. DEFINITION 8-1M. A *matrix* is an ordered rectangular array of numbers enclosed in parentheses. The individual numbers of the matrix are called its *elements* or *entries*.

In this book these entries are real numbers, and hereafter this is to be understood. Examples of matrices are

$$\begin{pmatrix}1\\3\end{pmatrix}, \quad (1\ 3\ -2), \quad \begin{pmatrix}1 & 3 & -2\\0 & 2 & 1\end{pmatrix}, \quad \begin{pmatrix}1 & 3 & -2\\3 & -2 & 1\\-2 & 1 & 3\end{pmatrix}.$$

DEFINITION 8-2M. The *order of a matrix* is indicated by a symbol such as 2×3 (read: "2 by 3") or more generally $n \times m$, where the first integer n indicates the number of rows and the integer m indicates the number of columns in the rectangular array.

A column vector is a matrix with only one column; a row vector is a matrix with only one row. If the number of rows equals the number of columns, the matrix is a *square matrix* and its order may be indicated by a single integer. A matrix may be designated by a single capital letter with its order shown as a subscript. Another convenient notation is to represent the matrix as a set of real numbers, using the same lower case letter with subscripts to indicate the position of the entry in the matrix. Thus a_{ij} represents the entry of a matrix which is in the ith row and the jth column, where i may have any value from 1 to n and j may have any value from 1 to m. Thus a 2×3 matrix might be represented in any of the following ways:

$$\mathbf{A} = \mathbf{A}_{2\times 3} = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\end{pmatrix} = (a_{ij}), \qquad (i = 1, 2;\ j = 1, 2, 3),$$

and a square matrix of order two might be represented by

$$\mathbf{B} = \mathbf{B}_2 = \begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{pmatrix} = (b_{ij}), \qquad (i, j = 1, 2).$$

Some of the symbols may be omitted if no misunderstanding is likely to occur.

DEFINITION 8–3M. Two matrices are *equal* if they contain the same number of rows and the same number of columns and if corresponding entries are equal.

Since equality of matrices depends only upon the properties of equality of real numbers, it follows that the laws of equality E1 to E4 are true for matrices as well as for real numbers.

The definition does not include two matrices which have different orders. For example,

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix},$$

the first being a 3×2 matrix and the second a 2×3 matrix. In this special case, the second matrix is obtained by interchanging the rows and columns of the first and is called the transpose of the first. The transpose of the matrix $\mathbf{A}$ is represented by the symbol $\mathbf{A}^T$ or by an interchange of the subscripts i and j; the transpose of (a_{ij}) being (a_{ji}).

The equality

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

implies $a = b_{11}$, $b = b_{12}$, $c = b_{21}$, and $d = b_{22}$;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

DEFINITION 8–4M. If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same order, their *sum*, indicated by $\mathbf{A} + \mathbf{B}$, is a matrix of the same order whose entries are the sums of the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$.

Addition for matrices of different orders is not defined.

EXAMPLE 8–4.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + (1 \quad 3)$$

is meaningless. In general,

$$\mathbf{A}_{n\times m} + \mathbf{B}_{n\times m} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}).$$

Since addition of real numbers is closed, commutative, and associative, it follows that these properties hold for matrices of the same order. If **A**, **B**, **C** are matrices of the same order, then

$$\mathbf{A} + \mathbf{B} \text{ is a matrix of the same order,} \tag{8–1M}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{8–2M}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C}. \tag{8–3M}$$

DEFINITION 8–5M. A *zero matrix* is one all of whose entries are 0.

A zero matrix is represented by the symbol $\mathbf{0}_{n\times m}$. The zero matrix has the important property

$$\mathbf{A} + \mathbf{0} = \mathbf{A}, \tag{8–4M}$$

where it is understood that **A** and **0** have the same order.

This is an immediate consequence of the properties of real numbers $a + 0 = a$. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

DEFINITION 8–6M. The *additive inverse* or the *negative of a matrix* **A** is the matrix whose entries are the negatives of the entries of **A**, and is indicated by $(-\mathbf{A})$.

If $\mathbf{A} = (a_{ij})$, then $(-\mathbf{A}) = (-a_{ij})$. It is easy to verify

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}. \tag{8–5M}$$

Matrices do not obey the Axioms of Order given for real numbers; hence positive and negative matrices are not defined. As indicated in Definition 8–6M, the *negative of a matrix* is defined.

The operation of *subtraction* of matrices of the same order, indicated by the use of the minus sign, $-$, is defined as follows:

DEFINITION 8–7M.

$$\text{If} \quad \mathbf{X} + \mathbf{A} = \mathbf{B}, \qquad \text{then} \qquad \mathbf{X} = \mathbf{B} - \mathbf{A}. \tag{8–6M}$$

If the entries of these matrices are x_{ij}, a_{ij}, b_{ij}, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$, respectively, then **X** is the matrix whose entries are the difference of the entries of **B** and **A**. Since $b_{ij} - a_{ij} = b_{ij} + (-a_{ij})$, where $(-a_{ij})$ is the additive inverse of a_{ij}, it follows that also

$$\mathbf{B} - \mathbf{A} = \mathbf{B} + (-\mathbf{A}), \tag{8–7M}$$

where $(-\mathbf{A})$ is the additive inverse of **A**.

THEOREM 8–1M.* *If* **A, B, C** *are matrices of the same order, and if* $\mathbf{A} = \mathbf{B}$, *then* $\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C}$; *and conversely, if* $\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C}$, *then* $\mathbf{A} = \mathbf{B}$.

Thus, all the fundamental laws of addition are valid for real matrices as well as for real numbers.

DEFINITION 8–8M. *The product of a number and a matrix* is the matrix whose entries are the entries of the given matrix each multiplied by the given number.

If the entry in the ith row and jth column is a_{ij}, then

$$k\mathbf{A} = \mathbf{A}k = k(a_{ij}) = (ka_{ij}),$$

where k is a real number.

Such multiplication is referred to as *scalar multiplication* or *numerical multiplication* to distinguish it from other types of multiplication which involve matrices.

As a consequence of this definition, it is seen that

$$0\mathbf{A} = \mathbf{0}; \qquad 1\mathbf{A} = \mathbf{A}; \qquad (-1)\mathbf{A} = (-\mathbf{A}) \tag{8–8M}$$

and that

$$\mathbf{B} - \mathbf{A} = \mathbf{B} + (-1)\mathbf{A}. \tag{8–9M}$$

The minus sign, $-$, has the same three interpretations with matrices that it has with real numbers.

THEOREM 8–2M. *If* k *and* l *are numbers and* **A** *and* **B** *are matrices, then*

$$(k + l)\mathbf{A} = k\mathbf{A} + l\mathbf{A}, \tag{8–10M}$$

$$k(l\mathbf{A}) = (kl)\mathbf{A} = l(k\mathbf{A}), \tag{8–11M}$$

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}. \tag{8–12M}$$

In summary, insofar as equality, addition, subtraction, and numerical multiplication are concerned, the algebra of matrices is like the algebra of real numbers.

Square matrices play a distinctive role. Associated with every square matrix is a determinant, but the two should not be confused. A square matrix is merely an ordered square array of numbers, whereas a determinant is an ordered square array of numbers to which a value is assigned. To distinguish between the two, parentheses are used to repre-

* The proof is left as an exercise.

sent matrices and vertical bars are used for determinants. If the square array of numbers is the same in both cases, the determinant is said to be associated with the matrix and is indicated by det A, where $\mathbf{A}$ is the matrix. Thus if

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and if

$$\mathbf{B} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \quad \text{then} \quad \det B = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

where the value of this determinant is obtained by the rules given in Section 5-7.

Note carefully that the value of a determinant may be zero, even when the associated matrix is not the zero matrix. For example,

$$\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \neq \mathbf{0}_{2\times 2}, \qquad \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = 0.$$

Problem Set 8-3

1. If

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & x+y \\ x-y & 5 \end{pmatrix},$$

find x and y.

2. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 4 & 2 \\ 1 & 0 \\ 2 & -4 \end{pmatrix},$$

compute (a) $\mathbf{A} + \mathbf{B}$; (b) $\mathbf{A} + \mathbf{B} + \mathbf{C}$; (c) $\mathbf{A} - \mathbf{B}$; (d) $\mathbf{B} - \mathbf{A}$; (e) $\frac{1}{2}\mathbf{A} - \frac{1}{3}\mathbf{C}$; (f) $\mathbf{X}$, if $3\mathbf{B} + \mathbf{X} = 2\mathbf{C}$.

3. Explain why it is not possible to compute the following sums.

(a) $\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} + (3\ 4)$ (b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 2$

4. If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same order, prove $\mathbf{A}^T + \mathbf{B}^T = (\mathbf{A} + \mathbf{B})^T$. First prove the statement for 2×2 matrices and then generalize.

5. Prove the Associative Law of Addition for matrices of the same order:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

6. Prove the Cancellation Law of Addition for matrices and its converse (Theorem 8-1M).

7. Prove (a) $k\mathbf{0} = \mathbf{0}$, (b) $0\mathbf{A} = \mathbf{0}$, (c) $\mathbf{A} + 0\mathbf{B} = \mathbf{A}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{0}$ are matrices of the same order.

8. Prove

$$(-1)\mathbf{A} = (-\mathbf{A}), \qquad \mathbf{B} - \mathbf{A} = \mathbf{B} + (-1)\mathbf{A},$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same order.

9. Prove Theorem 8-2M. State these laws in words.

10. Find the values of the determinants associated with each of the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 6 & 10 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & k & 7 \end{pmatrix}$ (d) $\mathbf{D} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & 0 & 7 \end{pmatrix}$

8–4 Vector multiplication. Experience has shown that different definitions of products of two vectors are useful for different purposes. Two such definitions are given here.

DEFINITION 8-9M. If $\mathbf{U}$ and $\mathbf{V}$ are *two row vectors* (or two *column vectors*) with the same number of components, then the *scalar or dot product* of $\mathbf{U}$ and $\mathbf{V}$, indicated by the use of a heavy center dot ($\mathbf{U}\cdot\mathbf{V}$), is the real number which is the sum of the products of corresponding components of the given vectors.

For example, $(u_1\ u_2\ u_3)\cdot(v_1\ v_2\ v_3) = u_1v_1 + u_2v_2 + u_3v_3$, and

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

is the same number.

Since multiplication of real numbers is commutative, it is easily verified that

$$\mathbf{V}\cdot\mathbf{U} = \mathbf{U}\cdot\mathbf{V}.$$

This special type of multiplication is commutative, but it is not associative. Indeed, the product $\mathbf{U}\cdot\mathbf{V}\cdot\mathbf{W}$ is meaningless, since $\mathbf{U}\cdot\mathbf{V}$ is a real number and the dot product is not defined for a scalar and a vector.

Another product which is closely related to matrices is defined as follows:

DEFINITION 8-10M: If $\mathbf{U}$ *is a row vector* and $\mathbf{V}$ is a *column vector* with the same number of components, then the product of $\mathbf{U}$ and $\mathbf{V}$, indicated by writing them in juxtaposition with $\mathbf{U}$ written first, is the vector which has a single component formed as the sum of the products of corresponding components of the given vectors.

For example:

$$(u_1\ u_2\ u_3)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (u_1v_1 + u_2v_2 + u_3v_3).$$

The product of a column vector and a row vector is not defined, and it is meaningless to talk about the commutative law of such a product. However, if **R** is a row vector and **C** is a column vector, then $\mathbf{C}^T$ (**C** transposed) and $\mathbf{R}^T$ are row and column vectors, respectively, and it follows that

$$\mathbf{RC} = \mathbf{C}^T\mathbf{R}^T = (r_1c_1 + r_2c_2 + \cdots + r_nc_n).$$

Since there is no essential difference between a real number and a vector with a single real component, the parentheses are usually omitted.

EXAMPLE 8–5.

(a) $(2\ 3)\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 4 - 3 = 1$; (b) $(2\ 3)\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 6 - 6 = 0$;

(c) $(2\ 3\ 4)\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 2 + 6 - 12 = -4$; (d) $(2\ 3\ 4)\begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = 0$.

Cases (b) and (d) show that the product may be zero even if neither the row nor column vector is a zero vector.

In case the product of two vectors $\mathbf{U}(u_1\ u_2)$ and $\mathbf{V}(v_1\ v_2)$ vanishes, the corresponding geometric vectors are perpendicular. If $\mathbf{U} = \overrightarrow{OA}$ and $\mathbf{V} = \overrightarrow{OB}$, the lines OA and OB are perpendicular if and only if the triangle OAB is a right triangle, that is,

$$AB^2 = OA^2 + OB^2.$$

In terms of coordinates (Fig. 8-8), $OA^2 = u_1^2 + u_2^2$, $OB^2 = v_1^2 + v_2^2$ and $AB^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2$. Hence

$$u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2,$$

which reduces to

$$u_1v_1 + u_2v_2 = 0.$$

The proof for three-space is very similar.

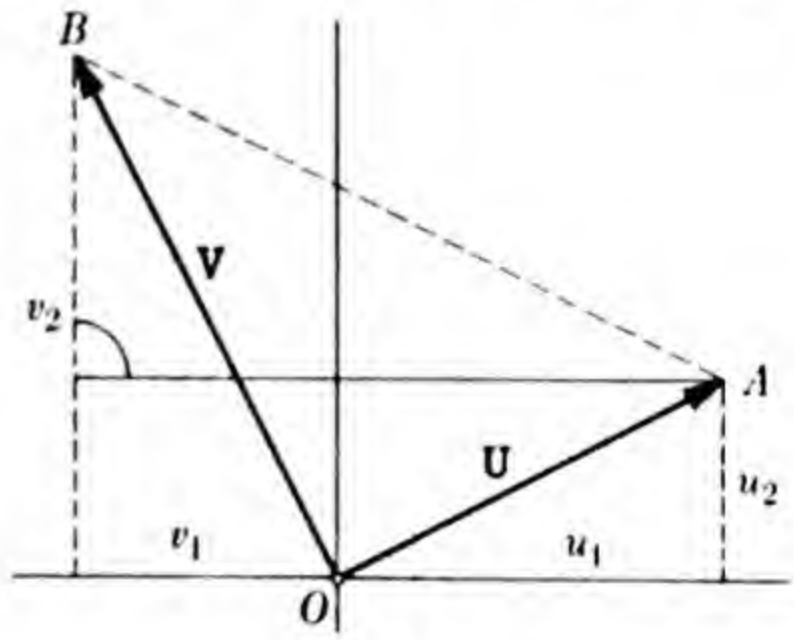


FIGURE 8–8

A linear equation can be written in terms of a vector product. The equation

$$ax + by + cz = d$$

is equivalent to

$$(a\ b\ c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = d.$$

If x, y, z indicate the quantities of three commodities and a, b, c represent their prices, then the budget equation can be written in vector form. Similarly, if q_1, q_2, q_3 represent the cost of producing one unit of each of three different commodities and x_1, x_2, x_3 represent the number of units produced, then the total cost, Q, given in vector form is

$$(q_1\ q_2\ q_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{Q}.$$

In such simple economic illustrations, the numbers involved are all positive (or zero) and the corresponding geometric vector is limited to the positive octant.

EXAMPLE 8–6. In a manufacturing process three commodities in amounts x, y, z are produced, and their prices p, q, r, respectively, are functions of the amounts produced, say $p = f(x)$, $q = g(y)$, $r = h(z)$. Find the total revenue received from the sale of these products.

Represent the prices by the row vector

$$\mathbf{P} = (p\ q\ r)$$

and the quantities by the column vector

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then the revenue R is given by

$$R = \mathbf{PX} = (p\ q\ r)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = px + qy + rz$$

$$= xf(x) + yg(y) + zh(z).$$

It may happen that the prices depend upon all three of the quantities x, y, z. For example, the commodities might be gasoline, kerosene, and

asphalt. In that case,

$$p = F(x, y, z), \qquad q = G(x, y, z), \qquad r = H(x, y, z),$$

and

$$R = xF + yG + zH = R(x, y, z)$$

is a joint revenue function.

8–5 Matrix multiplication. The definition of the product of a row vector and a column vector is used to define the product of two *compatible* matrices, that is, matrices which satisfy certain conditions on their orders. The matrix product of two matrices is defined if and only if their orders are of the form

$$n \times m \qquad \text{and} \qquad m \times k,$$

that is, provided the number of columns of the first equals the number of rows of the second. A row vector and a column vector with the same number of components are compatible. A 2×2 matrix is compatible with any $2 \times k$ matrix. A 2×3 matrix is compatible with any $3 \times k$ matrix. A square matrix of order n is compatible with a square matrix of the same order, with other matrices of n rows, and with column vectors with n components. A row vector of n components is compatible with any matrix of n rows.

Before the general definition of matrix multiplication is given, it is illustrated by two special cases. To find the product of a row vector **A** of three components and a 3×2 matrix, **M**, proceed as follows:

$$(a_1\ a_2\ a_3)\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix} = (\mathbf{AC}\ \mathbf{AD}).$$

Let **C** be the column vector whose components are the entries in the first column of **M**, and **D** be the column vector formed from the entries in the second column of **M**. The product is a vector of two components, the first being the vector product (Definition 8–10M) **AC** and the second the vector product **AD**.

EXAMPLE 8–7.

$$(2\ 3\ -1)\begin{pmatrix} 1 & 3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = (2 + 6 - 3 \quad 6 - 6 + 1) = (5\ 1).$$

This multiplication of "row by column" is the key to the general definition. To find the product of two 2×2 matrices, **M** and **N**,

$$\mathbf{MN} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} \mathbf{AC} & \mathbf{AD} \\ \mathbf{BC} & \mathbf{BD} \end{pmatrix},$$

let **A** be the row vector whose components are the entries of the first row of **M**, let **B** be the row vector formed from the elements of the second row of **M**. Let **C** and **D** be the column vectors formed from the entries in the first column and second column, respectively, of **N**. The product **MN** is a 2×2 matrix whose entries in the first row are the vector products **AC** and **AD** and in the second row are **BC** and **BD**.

EXAMPLE 8-8.

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2+2 & 4+6 \\ -4+1 & -8+3 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -5 \end{pmatrix}.$$

DEFINITION 8-11M. The *product* of an $n \times m$ matrix **M** and an $m \times k$ matrix **N** is an $n \times k$ matrix **P** obtained as follows: The entries in the first row of **P** are the vector products of the vector formed from the entries of the first row of **M** and the k column vectors formed from the entries in **N**; the entries in the second row of **P** are the vector products of the vector formed from the entries of the second row of **M** by these same column vectors, etc. In general, the entry p_{ij} of **P**, $(i = 1, 2, \ldots, n;$ $j = 1, 2, \ldots, k)$ is the vector product of the row vector whose components are the entries in the ith row of **M** and the column vector whose components are the entries in the jth column of **N**.

EXAMPLE 8-9.

$$\begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -6 & 1 \\ 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 8 - 18 + 12 & -4 + 3 + 20 \\ -4 - 12 & 2 + 2 \end{pmatrix} = \begin{pmatrix} 2 & 19 \\ -16 & 4 \end{pmatrix}.$$

The product of a (2×3) and a (3×2) matrix is a (2×2) matrix.

A system of equations

$$\begin{aligned} a_1x + a_2y + a_3z &= d_1 \\ b_1x + b_2y + b_3z &= d_2 \\ c_1x + c_2y + c_3z &= d_3 \end{aligned}$$

can be written in the condensed form

$$\mathbf{AX} = \mathbf{D},$$

where **A** is the matrix of coefficients, **X** is a column vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{and } \mathbf{D} \text{ is the column vector } \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

An equivalent form is

$$\mathbf{X}^T\mathbf{A}^T = \mathbf{D}^T,$$

where T indicates the transpose, so that $\mathbf{X}^T$ and $\mathbf{D}^T$ are row vectors. In detail,

$$(x\ y\ z)\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (d_1\ d_2\ d_3).$$

EXAMPLE 8–10. Consider the following greatly oversimplified production problem. A manufacturer of furniture wishes to produce 40 tables, 120 chairs, 20 desks, and 10 book cases. It requires 8 units of material and 3 man-hours of labor for each table; 4 units of material and 2 man-hours of labor for each chair; 10 units of material and 6 man-hours of labor for each desk; and 12 units of material and 10 man-hours of labor for each book case. The material costs \$.75 per unit and labor is \$3.00 per man-hour. Find the total cost as follows: Consider the quantities to be produced as a row vector, the amount of materials and labor as a matrix, and the costs as a column vector.

$$\text{Total cost} = (40\ 120\ 20\ 10)\begin{matrix} \text{Material} & \text{Labor} \\ \begin{pmatrix} 8 & 3 \\ 4 & 2 \\ 10 & 6 \\ 12 & 10 \end{pmatrix} \end{matrix}\begin{pmatrix} \frac{3}{4} \\ 3 \end{pmatrix}$$

$$= \begin{matrix} \text{Material} & \text{Labor} \\ (1220 & 580) \end{matrix}\begin{pmatrix} \frac{3}{4} \\ 3 \end{pmatrix} = \$2655.$$

Note that the product of the first vector by the so-called activity matrix gives the total material and labor used, whereas the final product is the total cost.

PROBLEM SET 8–4

1. If

$$\mathbf{U} = (2\ 3), \quad \mathbf{V} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad \mathbf{W} = (2\ 3\ 1), \quad \mathbf{X} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix},$$

determine, if possible, the following vector products.

(a) **UV** (b) **VU** (c) **WX** (d) **WY** (e) **XY**

2. If $A(2, 3)$, $B(-6, 4)$, $C(3, 2)$, $D(-6, 9)$, which of the six possible pairs of geometric vectors from $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ are perpendicular?

3. If $A(2, 1, -3)$, $B(2, 1, 1)$, $C(-1, 5, 1)$, $D(2, -4, 0)$, which of the six possible pairs of geometric vectors from $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ are perpendicular?

4. In a manufacturing process, three commodities in amounts x, y, z are produced, and their prices p, q, r are given by the demand laws $p = 12 - 4x$, $q = 16 - 3y$, $r = 10 - 5z$. Write the total revenue R as a vector product and compute R as a function of x, y, z. State the limitations that must be imposed upon x, y, z.

5. Perform the following matrix multiplications, when possible.

(a) $(2\ 3)\begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -2 \end{pmatrix}$ (b) $(2\ -1\ 3)\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}$

(c) $(2\ -1\ 3)\begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -2 \end{pmatrix}$

6. Perform the following matrix multiplications, when possible.

(a) $\begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}(2\ 3)$

(c) $\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

7. If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix},$$

compute (a) **AB**, (b) **[AB]C**, (c) **BC**, (d) **A[BC]**, (e) **A[B + C]** and verify that it equals **AB + AC**, (f) **[B + C]A** and verify that it equals **BA + CA**.

8. If

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -2 & -4 \\ 3 & 2 \end{pmatrix},$$

verify that **AB** = **AC**.

9. If

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix},$$

verify that **AB** = **0**.

10. If $\mathbf{A}^2$ means $\mathbf{AA}$, show that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2, \quad \begin{pmatrix} 0 & k \\ 1/k & 0 \end{pmatrix}^2,$$

where k is any nonzero number, are all the same.

11. Write the following systems of equations in matrix form (a) involving column vectors; (b) involving row vectors.

(a) $3x + 2y = 7$
$-2x + y = 2$

(b) $3x + 2y = 7$
$-2x + y = 2$
$5x + y = 5$

(c) $3x + 2y - z = 9$
$-2x + y + 3z = 13$

(d) $3x + 2y - z = 9$
$-2x + y + 3z = 13$
$5x + 3y - 2z = 6$

12. A contractor agrees to manufacture 5 units of article A, 15 units of B, 3 units of C, 7 units of D. Each unit of A requires 2 man-hours of labor, 3 units of material, and 2 hours use of special equipment; each unit of B requires 3 man-hours of labor, 1 unit of material, and 1 hour use of special equipment; each unit of C requires 4 man-hours of labor, 2 units of material; each unit of D requires 1 man-hour of labor, 4 units of material, and 2 hours use of special equipment. Each man-hour of labor costs \$3, each unit of material costs \$1, and each hour use of special equipment costs \$4. Find the total cost by considering the quantities to be produced as a row vector; the amounts of labor, material, and special equipment as the activity matrix; and the costs as a column vector.

8-6 Properties of matrix multiplication. Some, but not all, of the laws of multiplication for real numbers remain valid for matrix multiplication.

Closure. The product of two matrices is defined if and only if the matrices are compatible. *Two square matrices of the same order are always compatible.*

Commutative law. In general, the product of two matrices is not commutative. If the matrices **A** and **B** have orders $n \times m$ and $m \times k$, so that **AB** is defined, the commutative property may fail for any one of the following reasons:

(1) $n \neq k$, **BA** is not defined;

(2) $n = k \neq m$, **AB** and **BA** are square matrices of different orders;

(3) When **A** and **B** are square matrices of order n, the entries in the products may differ. For example,

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -3 & -5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 8 \\ -5 & 5 \end{pmatrix}.$$

EXAMPLE 8–11. Determine, if possible, the number x, so that the matrices

$$\mathbf{X} = \begin{pmatrix} 3 & 1 \\ x & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

are commutative.

$$\mathbf{XY} = \begin{pmatrix} 9 & 5 \\ 2x + 9 & x + 6 \end{pmatrix}, \qquad \mathbf{YX} = \begin{pmatrix} x + 6 & 5 \\ 2x + 9 & 9 \end{pmatrix}.$$

These products are equal if and only if $x + 6 = 9$, or $x = 3$. For any value of $x \neq 3$, $\mathbf{XY} \neq \mathbf{YX}$.

Associative law. If matrices **A**, **B**, **C** *are compatible for the product* **ABC**, then

$$[\mathbf{AB}]\mathbf{C} = \mathbf{A}[\mathbf{BC}].*$$

A proof is given for the case of 2×2 matrices. The proof for the general case is similar, the difficulties being related to notation rather than to procedures. The case where **A** is a row vector and **C** is a column vector is considered first.

$$[\mathbf{AB}]\mathbf{C} = \left[(a_1 \; a_2)\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (a_1b_1 + a_2b_3 \quad a_1b_2 + a_2b_4)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_2 + a_2b_4c_2.$$

$$\mathbf{A}[\mathbf{BC}] = (a_1 \; a_2)\left[\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right] = (a_1 \; a_2)\begin{pmatrix} b_1c_1 + b_2c_2 \\ b_3c_1 + b_4c_2 \end{pmatrix}$$

$$= a_1b_1c_1 + a_1b_2c_2 + a_2b_3c_1 + a_2b_4c_2.$$

These are the same numbers. If now

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_1' & a_2' \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_1 & c_1' \\ c_2 & c_2' \end{pmatrix},$$

then $[\mathbf{AB}]\mathbf{C}$ and $\mathbf{A}[\mathbf{BC}]$ are determined by computations similar to those above with $(a_1 \; a_2)$ replaced by $(a_1' \; a_2')$,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} c_1' \\ c_2' \end{pmatrix}.$$

The procedure for matrices of higher order is similar.†

---

* Square brackets are used as signs of grouping, since parentheses are used in the matrix notation.

† For a proof in the general case, involving the summation notation, see Mathematics for High School, *Introduction to Matrix Algebra*, School Mathematics Study Group, 1960.

Distributive laws. If **A** *and* **B** *are compatible (for multiplication), and if* **B** *and* **C** *are conformable (for addition), then*

$$\mathbf{A}[\mathbf{B} + \mathbf{C}] = \mathbf{AB} + \mathbf{AC}; \qquad [\mathbf{B} + \mathbf{C}]\mathbf{A} = \mathbf{BA} + \mathbf{CA}.$$

Since matrix multiplication is not commutative, the order of writing the symbols must be preserved. If **A** is a row vector, $(a_{11}\ a_{12}\ a_{13})$, and **B** and **C** are 3×2 matrices, then

$$\mathbf{AB} = (a_{11}\ a_{12}\ a_{13}) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{33} \end{pmatrix},$$

$$\mathbf{AC} = (a_{11}\ a_{12}\ a_{13}) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}$$

and

$$\mathbf{A}[\mathbf{B} + \mathbf{C}] = (a_{11}\ a_{12}\ a_{13}) \begin{pmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + b_{21} & b_{22} + b_{22} \\ b_{31} + b_{31} & b_{32} + b_{32} \end{pmatrix}.$$

It is straightforward computation to verify that

$$\mathbf{A}[\mathbf{B} + \mathbf{C}] = \mathbf{AB} + \mathbf{AC}.$$

If **A** has more than one row, a change in the subscripts to $(a_{i1}\ a_{i2}\ a_{i3})$ gives similar results. The procedure in the most general case is similar.

To prove $[\mathbf{B} + \mathbf{C}]\mathbf{A} = \mathbf{BA} + \mathbf{CA}$, first consider **A** as a column vector, and then as a set of column vectors.

Although matrix multiplication is distributive with respect to addition, the failure of the commutative law means that the usual formulas for real numbers concerning special products and factoring are not, in general, valid for matrices.

$$[\mathbf{A} + \mathbf{B}]^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2,$$

$$[\mathbf{A} + \mathbf{B}][\mathbf{A} - \mathbf{B}] = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2,$$

for any **A** and **B** which are square matrices of the same order.

Identity matrix. If **A** *is a matrix of order* $n \times m$*, then there exists a square matrix* $\mathbf{I}_n$ *and a square matrix* $\mathbf{I}_m$ *such that* $\mathbf{I}_n\mathbf{A} = \mathbf{A}$ and $\mathbf{AI}_m = \mathbf{A}$. *If* **A** *is a square matrix of order* n*, then* $\mathbf{I}_n = \mathbf{I}_m$*, that is,*

$$\mathbf{I}_n\mathbf{A}_n = \mathbf{A}_n = \mathbf{A}_n\mathbf{I}_n.$$

The matrices $\mathbf{I}_n$ and $\mathbf{I}_m$ are called *identity matrices* or *unit matrices.* They are matrices in which entries on the main diagonal are 1 and all other entries are 0.

EXAMPLE 8–12.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Multiplicative inverse. If $\mathbf{A}$ *is a matrix of order* $n \times m$ *and* $\mathbf{X}$ *is a matrix of order* $m \times n$ *such that* $\mathbf{AX} = \mathbf{I}_n$, *then* $\mathbf{X}$ *is called a right inverse of* $\mathbf{A}$ *and is designated by the symbol* $\mathbf{A}^{-1}$.

If $\mathbf{Y}$ *is a matrix of order* $m \times n$ *such that* $\mathbf{YA} = \mathbf{I}_m$, *then* $\mathbf{Y}$ *is called a left inverse* of $\mathbf{A}$. *The symbol* $\mathbf{A}^{-1}$ *is used for* $\mathbf{Y}$ *also, subscripts being used if necessary to distinguish* $\mathbf{X}$ *from* $\mathbf{Y}$.

If $\mathbf{A}$ *is a square matrix of order* n, *and if there exists a matrix* $\mathbf{A}^{-1}$ *such that*

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1},$$

then $\mathbf{A}^{-1}$ *is called the inverse of* $\mathbf{A}$.

The existence and uniqueness of $\mathbf{A}^{-1}$ for a square matrix is discussed in the next section. The general situation is now illustrated.

EXAMPLE 8–13. It may be verified that

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $\mathbf{A}$ is a left inverse of $\mathbf{B}$ and $\mathbf{B}$ is a right inverse of $\mathbf{A}$.

It may be verified that

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix},$$

so that

$$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \quad \text{is the inverse of} \quad \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

Zero matrix and multiplication. The zero matrix $\mathbf{0}$ was defined in terms of addition, but if $\mathbf{0}$ and $\mathbf{A}$ are compatible, it is seen that $\mathbf{0A} = \mathbf{0}$. The converse, however, is *not true*. It is possible to have two nonzero matrices whose product is a zero matrix. For example,

$$\mathbf{AB} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

A characteristic of the matrix **A** above is that its associated determinant is zero:

$$\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 6 - 6 = 0.$$

Cancellation for multiplication. It is easy to prove on the basis of the Substitution Principle, which is valid for matrices, that if $\mathbf{A} = \mathbf{B}$, then $\mathbf{AC} = \mathbf{BC}$. The converse, even when $\mathbf{C} \neq \mathbf{0}$, is not true. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix} \neq \mathbf{A}, \quad \mathbf{C} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix},$$

then

$$\mathbf{AC} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ -6 & -9 \end{pmatrix},$$

$$\mathbf{BC} = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 15 \\ -6 & -9 \end{pmatrix} = \mathbf{AC},$$

but $\mathbf{A} \neq \mathbf{B}$. Note that $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$.

Similarly, the left cancellation law is not valid. If $\mathbf{CA} = \mathbf{CB}$, and $\mathbf{C} \neq \mathbf{0}$, it is not possible to conclude that $\mathbf{A} = \mathbf{B}$.

The *failure* of the *commutative law*, of the *law of cancellation*, and of the *zero law*—if $\mathbf{AB} = \mathbf{0}$, then either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$—are three of the main differences between the algebra of matrices and the algebra of real numbers. These laws are closely related to the existence of a multiplicative inverse.

Problem Set 8–5

1. (a) If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix},$$

determine if $\mathbf{AB} = \mathbf{BA}$.

(b) If

$$\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix},$$

determine whether $\mathbf{CD} = \mathbf{DC}$.

2. Determine, if possible, values of x and y so that if

(a) $\mathbf{A} = \begin{pmatrix} x & 2 \\ -2 & y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, then $\mathbf{AB} = \mathbf{BA}$;

(b) $\mathbf{A} = \begin{pmatrix} x & 2 \\ -1 & y \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, then $\mathbf{AB} = \mathbf{BA}$.

3. Prove the associative law $[\mathbf{AB}]\mathbf{C} = \mathbf{A}[\mathbf{BC}]$ if

(a) $\mathbf{A}$ is a 1×3 vector, $\mathbf{B}$ is a 3×2 matrix, and $\mathbf{C}$ is a 2×1 vector;

(b) $\mathbf{A}$ is a 1×3 vector, $\mathbf{B}$ is a 3×3 matrix, and $\mathbf{C}$ is a 3×1 vector.

4. Prove the distributive law $[\mathbf{B} + \mathbf{C}]\mathbf{A} = \mathbf{BA} + \mathbf{CA}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are square matrices of order 2.

5. If

$$\mathbf{A} = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix},$$

write the left and right identity matrices for $\mathbf{A}$ and $\mathbf{B}$.

6. (a) Verify that

$$\begin{pmatrix} 4/5 & -1/5 \\ -3/5 & 2/5 \end{pmatrix} \quad \text{is the inverse of} \quad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

(b) Verify that

$$\begin{pmatrix} 6 & -9 & 4 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{pmatrix} \quad \text{is the inverse of} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & -3 & 0 \end{pmatrix}.$$

(c) Verify that

$$\begin{pmatrix} 1 & -3 & -2 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{is a left inverse of} \quad \begin{pmatrix} 1 & 2 \\ -2 & 0 \\ 3 & 1 \end{pmatrix}.$$

7. If

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

show that $\mathbf{AB} = 0$ but $\mathbf{BA} \neq 0$.

8. If

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix},$$

show that $\mathbf{AB} = 0 = \mathbf{BA}$.

9. If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix},$$

show that $\mathbf{AB} = 0 \neq \mathbf{BA}$.

10. (a) Prove that if $\mathbf{A} = \mathbf{B}$, then $\mathbf{AC} = \mathbf{BC}$.

(b) If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & 4 \\ 3 & -3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix},$$

show that $\mathbf{AC} = \mathbf{BC}$, $\mathbf{A} \neq \mathbf{B}$.

11. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are square matrices and if $\mathbf{C}$ has a multiplicative inverse $\mathbf{C}^{-1}$, and if (a) $\mathbf{AC} = \mathbf{BC}$, or if (b) $\mathbf{CA} = \mathbf{CB}$, prove $\mathbf{A} = \mathbf{B}$.

8-7 Matrices and linear equations. It was pointed out in Section 8-5 that the system of equations

$$\begin{aligned} a_1x + a_2y + a_3z &= d_1 \\ b_1x + b_2y + b_3z &= d_2 \\ c_1x + c_2y + c_3z &= d_3 \end{aligned} \tag{8-17}$$

is equivalent to the matrix equation

$$\mathbf{AX} = \mathbf{D}, \tag{8-18}$$

where $\mathbf{A}$ is the matrix of the coefficients, $\mathbf{X}$ and $\mathbf{D}$ are the column vectors $(x\ y\ z)^T$ and $(d_1\ d_2\ d_3)^T$. Similarly, any system of n linear equations in m unknowns can be written in the form (8-18), where $\mathbf{A}$ is an $n \times m$ matrix and $\mathbf{X}$ and $\mathbf{D}$ are column vectors with m components. If $\mathbf{A}$ has a left inverse $\mathbf{A}^{-1}$, then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{D}$ or

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{D}. \tag{8-19}$$

THEOREM 8-7. *If* $\mathbf{A}$ *is a square matrix of order* 2 *or order* 3, *there exists a unique inverse of* $\mathbf{A}$, *provided* $\det A \neq 0$.

The proof gives a procedure for finding $\mathbf{A}^{-1}$. First suppose that $\mathbf{A}$ is a 2×2 matrix. Seek numbers u_1, u_2, v_1, v_2 so that

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix equation is equivalent to two pairs of equations in the unknowns:

$$\begin{cases} a_1u_1 + b_1u_2 = 1 \\ a_2u_1 + b_2u_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_1v_1 + b_1v_2 = 0 \\ a_2v_1 + b_2v_2 = 1 \end{cases}. \tag{8-20}$$

There will be a unique solution $(u_1\ u_2)$ and $(v_1\ v_2)$ if and only if the de-

terminant Δ associated with the matrix **A** is different from zero:

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0.$$

In that case,

$$u_1 = \frac{\begin{vmatrix} 1 & b_1 \\ 0 & b_2 \end{vmatrix}}{\Delta} = \frac{b_2}{\Delta}, \qquad u_2 = \frac{\begin{vmatrix} a_1 & 1 \\ a_2 & 0 \end{vmatrix}}{\Delta} = \frac{-a_2}{\Delta},$$

$$v_1 = \frac{\begin{vmatrix} 0 & b_1 \\ 1 & b_2 \end{vmatrix}}{\Delta} = \frac{-b_1}{\Delta}, \qquad v_2 = \frac{\begin{vmatrix} a_1 & 0 \\ a_2 & 1 \end{vmatrix}}{\Delta} = \frac{a_1}{\Delta}.$$

Further,

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix} \frac{1}{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \frac{1}{\Delta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

since $a_1b_2 - a_2b_1 = \Delta$. Hence

$$\frac{1}{\Delta} \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}$$

is the inverse of **A**, namely, $\mathbf{A}^{-1}$. The matrix $\mathbf{A}^{-1}$ is unique, and Eq. (8–19) gives a formal way of solving the equations in two unknowns corresponding to Eqs. (8–17).

If $\Delta = 0$, both sets of equations in Eq. (8–20) are inconsistent and **A** has no inverse.

Similarly,

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is equivalent to

$$\begin{aligned} a_1u_1 + b_1u_2 + c_1u_3 &= 1, \\ a_2u_1 + b_2u_2 + c_2u_3 &= 0, \\ a_3u_1 + b_3u_2 + c_3u_3 &= 0, \end{aligned} \tag{8–21}$$

and two other equations which in matrix form are

$$\begin{aligned} \mathbf{A}^T\mathbf{V} &= (0\ 1\ 0), \\ \mathbf{A}^T\mathbf{W} &= (0\ 0\ 1). \end{aligned} \tag{8–22}$$

There will be unique $(u_1\ u_2\ u_3)$, $(v_1\ v_2\ v_3)$, $(w_1\ w_2\ w_3)$ if and only if

det $A \neq 0$. In that case

$$u_1 = \frac{\begin{vmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{A_1}{\Delta},$$

where A_1 is the cofactor (signed minor) of a_1 in the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ \mathrm{b}_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

which is the transpose of Δ. Similarly,

$$u_2 = \frac{\begin{vmatrix} a_1 & 1 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix}}{\Delta} = \frac{B_1}{\Delta}; \qquad u_3 = \frac{\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}}{\Delta} = \frac{C_1}{\Delta}$$

and

$$(v_1\ v_2\ v_3) = \frac{1}{\Delta}(A_2\ B_2\ C_2), \qquad (w_1\ w_2\ w_3) = \frac{1}{\Delta}(A_3\ B_3\ C_3).$$

Further,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}\frac{1}{\Delta} = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}\frac{1}{\Delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

since

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3,$$
$$0 = a_1B_1 + a_2B_2 + a_3B_3,$$

etc. (See Eqs. (5–12) and (5–18) of Section 5–7.) Hence

$$\mathbf{A}^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \text{ is the inverse of } \mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$.

If $\Delta = 0$, then Eq. (8–21) and those that correspond to Eq. (8–22) are inconsistent unless all the cofactors are zero. But $\mathbf{0A} \neq \mathbf{I}$. Hence $\mathbf{A}$ has no inverse.

EXAMPLE 8–14. Solve the system of equations

$$\begin{aligned} 2x - 4y + 3z &= 3 \\ 4x - 6y + 5z &= 2 \\ -2x + y - z &= 1 \end{aligned}$$

by finding the inverse of the matrix of coefficients.

$\Delta = \det A^T$

$$= \begin{vmatrix} 2 & 4 & -2 \\ -4 & -6 & 1 \\ 3 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -2 \\ -3 & -4 & 1 \\ 2 & 3 & -1 \end{vmatrix} = (-2)(-1) = 2 \neq 0.$$

The inverse $\mathbf{A}^{-1}$ is obtained by forming the matrix whose entries are the cofactors of $\det A^T/2$. Hence

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{pmatrix}.$$

If the given equations are

$$\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

then

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

so that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -4 \\ -4 \end{pmatrix}.$$

This method of finding the inverse of **A** is fundamentally that of solving the equations by means of determinants, except for formal details. There is another method of finding the inverse which is related to the method of successive elimination (Section 5–8). In this, the given system is replaced by an equivalent system by means of the permissible operations of multiplying by a constant or combining equations by addition or subtraction. These operations are performed until all coefficients except those

that correspond to the entries along the principal diagonal become zero. If the various sets of equivalent equations are written in matrix form, these operations can be performed on the matrices and the system reduced to $(x\ y\ z)^T = (x_0\ y_0\ z_0)^T$. If the given system is the system (8-17), it is written in the form

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

If the first of the equations is multiplied by k and the others left unchanged, it is seen that the new system is

$$\begin{pmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

If the first equation is now added to the second and the others remain unchanged, the new system is

$$\begin{pmatrix} ka_1 & ka_2 & ka_3 \\ b_1 + ka_1 & b_2 + ka_2 & b_3 + ka_3 \\ c_2 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

In the method of successive elimination, k is selected so $b_1 + ka_1 = 0$; the process is repeated to eliminate x in the third equation; this is followed by the elimination of y from the third equation, and finally the elimination of all terms except those that involve y for the second equation and those that involve x for the first equation. If the operations are performed on the matrix which is the coefficient of

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and then the same operations are performed on the matrix which is the coefficient of

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

the following equation results:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{B} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Since

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

and when $\det A \neq 0$, there is a unique inverse, $\mathbf{B} = \mathbf{A}^{-1}$.

The advantage of this method is that it can be used to replace the given system by a simpler system even if $\det A = 0$, or if there are n equations in m variables, whether $n = m$, $n < m$, or $n > m$.

If $\mathbf{A}^{-1}$ is not wanted, but only the solution vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is desired, the different operations are applied to the column vector

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

EXAMPLE 8–15. Solve the system of equations given in Example 8–14 by the method of successive elimination (sometimes called the *diagonalization method*), writing the equations in matrix form.

$$\begin{pmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Step 1. Multiply the first row by -2 and add to the second row.

Step 2. Add the first row to the third row.

$$\begin{pmatrix} 2 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Step 3. Multiply the second row by 3 and the third row by 2.

Step 4. Add the second row to the third row.

$$\begin{pmatrix} 2 & -4 & 3 \\ 0 & 6 & -3 \\ 0 & -6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -6 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & -4 & 3 \\ 0 & 6 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -6 & 3 & 0 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Step 5. Add 3 times the third row to the second.
Step 6. Divide the second row by 6.
Step 7. Subtract 3 times the third row from the first row and then add 4 times the second row to this.

$$\begin{pmatrix}2 & -4 & 3\\ 0 & 6 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ -18 & 12 & 6\\ -4 & 3 & 2\end{pmatrix}\begin{pmatrix}3\\ 2\\ 1\end{pmatrix};$$

$$\left(\begin{array}{ccc}2 & -4 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right)\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\left(\begin{array}{ccc|c}1 & 0 & 0 & 3\\ -3 & 2 & 1 & 2\\ -4 & 3 & 2 & 1\end{array}\right)\begin{array}{c}1\\ 4;\\ -3\end{array}$$

$$\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1 & -1 & -2\\ -3 & 2 & 1\\ -4 & 3 & 2\end{pmatrix}\begin{pmatrix}3\\ 2\\ 1\end{pmatrix}.$$

And finally,
Step 8. Divide the first row by 2.

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & -1\\ -3 & 2 & 1\\ -4 & 3 & 2\end{pmatrix}\begin{pmatrix}3\\ 2\\ 1\end{pmatrix}.$$

This agrees with the previous result, so that $(x\ y\ z)$ is $(-\frac{1}{2}\ -4\ -4)$. If the various steps are performed on the column vector to the right, it takes the successive forms indicated below, ending with the solution vector:

$$\begin{pmatrix}3\\ 2\\ 1\end{pmatrix},\quad \begin{pmatrix}3\\ -4\\ 1\end{pmatrix},\quad \begin{pmatrix}3\\ -4\\ 4\end{pmatrix},\quad \begin{pmatrix}3\\ -12\\ 8\end{pmatrix},\quad \begin{pmatrix}3\\ -12\\ -4\end{pmatrix},$$

$$\begin{pmatrix}3\\ -24\\ -4\end{pmatrix},\quad \begin{pmatrix}3\\ -4\\ -4\end{pmatrix},\quad \begin{pmatrix}-1\\ -4\\ -4\end{pmatrix},\quad \begin{pmatrix}-\frac{1}{2}\\ -4\\ -4\end{pmatrix}.$$

The inverse matrix is not written.

EXAMPLE 8–16. Solve the system of equations

$$\begin{aligned}2x - 4y + 3z &= 3,\\ 4x - 6y + 5z &= 2,\\ x - y + z &= k\end{aligned}$$

by writing the equations in matrix form and then using the diagonalization method.

One procedure is:

$$\begin{pmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix},$$

$$\begin{pmatrix} 0 & -2 & 1 \\ 0 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix}.$$

This shows that the determinant of the coefficients is zero, so that the equations are either dependent or inconsistent. The first row shows that

$$0 = 3 - 2 + 2k, \qquad \text{or} \qquad k = -\tfrac{1}{2},$$

if there is any solution. If $k \neq -\frac{1}{2}$ there is no solution. Suppose $k = -\frac{1}{2}$ and consider x as arbitrary. The second and third equations equivalent to this matrix equation can be written in matrix form as follows:

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -\frac{1}{2} - x \end{pmatrix}.$$

Then

$$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -\frac{1}{2} - x \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ -\frac{1}{2} - x \end{pmatrix}.$$

Hence $y = -\frac{9}{2} - x$, $z = -5 - 2x$, x arbitrary.

The same set of equations could be solved by considering either y or z as arbitrary.

Problem Set 8–6

1. Solve the following matrix equations $\mathbf{AX} = \mathbf{D}$ by finding $\mathbf{A}^{-1}$, using the cofactor method.

(a) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -5 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

2. Solve the following matrix equations $\mathbf{AX} = \mathbf{D}$ by finding $\mathbf{A}^{-1}$, using the cofactor method.

(a) $\begin{pmatrix} 2 & -2 & 1 \\ -2 & 3 & 1 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 5 \\ 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -8 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 3 & -2 \\ 1 & -1 & 5 \\ -1 & 2 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$

3. Solve each of the equations in problem 1 by the diagonalization method (equivalent to the method of successive elimination).

4. Solve each of the equations in problem 2 by the diagonalization method (equivalent to the method of successive elimination). Compare $\mathbf{A}^{-1}$ found by this method with the values found in problem 2.

5. Solve each of the following sets of equations by one of the methods of this section, considering x as arbitrary. Repeat the solution considering z as arbitrary.

(a) $2x + 3y - 4z = 21$
$3x - 4y + 5z = -16$

(b) $3x + 4y + 5z = 6$
$5x + 2y - z = 4$

6. Solve Example 8-16, when $k = -\frac{1}{2}$, considering

(a) z as arbitrary; (b) y as arbitrary.

7. Determine k so that the following matrix equations have a solution and find the solutions.

(a) $\begin{pmatrix} 2 & 3 & -4 \\ 1 & 1 & -1 \\ 4 & 5 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ 2 \\ 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 3 & -4 \\ 1 & 2 & -3 \\ 1 & 4 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ k \\ 1 \end{pmatrix}$

8. Use matrix methods to show that

$$\begin{pmatrix} 2 & 3 \\ -3 & 1 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 37 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 9 \\ 37 \end{pmatrix}$$

has a solution. Replace the three equations by two equations in matrix form and find the solution.

9. Complete the details of the solution which proves that

$$\frac{1}{\Delta}\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \quad \text{is the inverse of} \quad \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

10. Show that the vector equation $\mathbf{AX} = \mathbf{0}$ or

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where det $A = 0$ and the cofactor $C_1 \neq 0$, is equivalent to

$$\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -a_1x \\ -a_2x \end{pmatrix},$$

and that $x:y:z = C_1:C_2:C_3$.

8–8 Linear inequalities in two and three variables. A fundamental postulate of plane geometry which is essential for the logical development of Euclidean plane geometry, is the following.*

PLANE SEPARATION POSTULATE. *A line lying in a given plane separates the plane into two regions, called half-planes, each of which is convex.*

The term "separates" means that if P and Q are points in different half-planes, then the segment $\overline{PQ}$ contains a point of the line. The term "convex" means that if P_1 and P_2 are two points in the same half-plane, then every point of the segment $\overline{P_1P_2}$ lies in that half-plane.

The algebraic equivalent of this postulate is the following:

THEOREM 8–8. *The line $ax + by + c = 0$ separates the xy-plane into two regions, called half-planes, such that if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie in different half-planes, then the numbers $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ have opposite signs; and if P_1 and P_2 lie in the same half-plane, then these numbers have the same sign.*

In view of the Law of Trichotomy, this is equivalent to saying that P_1 and P_2 lie in different half-planes if and only if $(ax_1 + by + c)$ and $(ax_2 + by_2 + c)$ have opposite signs. One half-plane consists of all points (x, y) such that $ax + by + c > 0$, and the other half-plane consists of all points such that $ax + by + c < 0$. Points on the line do not belong to either half-plane; the line is called the *boundary* of either half-plane. To represent a half-plane and its boundary, the notation $ax + by + c \geqq 0$ (or $\leqq 0$) is used.

Proof of Theorem 8–8. The proof depends upon Theorem 8–5 involving geometric vectors. If the line segment $\overline{P_1P_2}$ contains the point $P(x, y)$ on the line $ax + by + c = 0$ (Fig. 8–9), Theorem 8–5 becomes

$$(x\ y) = t(x_1\ y_1) + (1 - t)(x_2\ y_2), \qquad (0 < t < 1),$$

which is equivalent to

$$x = tx_1 + (1 - t)x_2$$
$$y = ty_1 + (1 - t)y_2.$$

---

* See the School Mathematics Study Group text on geometry.

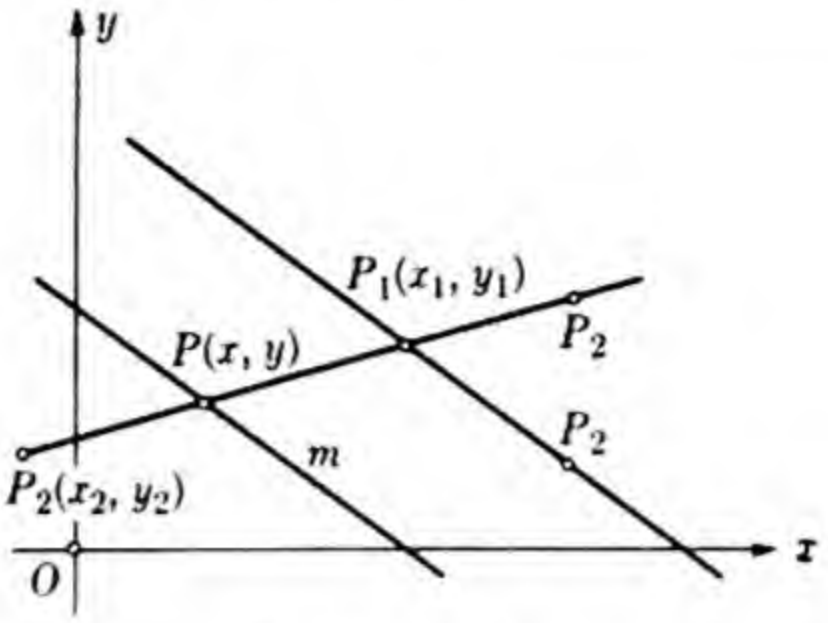


FIGURE 8-9

If the first of these equations is multiplied by a, the second by b, and the results added to c, then

$$(ax + by + c) = t(ax_1 + by_1 + c) + (1 - t)(ax_2 + by_2 + c), \quad (8\text{-}23)$$

for $c = tc + (1 - t)c$. Since t and $1 - t$ are both positive and $ax + by + c = 0$, it follows that $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ have opposite signs.

Now suppose P_1 and P_2 lie in the same half-plane so that the segment $\overline{P_1P_2}$ contains no point of the given line m. If the *line* P_1P_2 is parallel to the given line, its equation has the form $ax + by + c = d$, and $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ both are equal to d and hence have the same sign. If the *line* P_1P_2 meets m in P, then one of P_1 and P_2 is between P and the other. Suppose P_1 is between P and P_2 (Fig. 8-9). Then, as before,

$$(ax_1 + by_1 + c) = t(ax + by + c) + (1 - t)(ax_2 + by_2 + c), \quad (0 < t < 1). \quad (8\text{-}24)$$

Since $ax + by + c = 0$ and $1 - t$ is positive, it follows that

$$(ax_1 + by + c) \quad \text{and} \quad (ax_2 + by_2 + c)$$

have the same sign.

It follows by the method of contradiction that P_1 and P_2 lie in different half-planes if and only if $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ have opposite signs, and lie in the same half-plane if and only if these numbers have the same sign. To determine for which half-plane the sign is $+$, it is only necessary to try a single point. If the line does not pass through the origin, the simplest point to use is the origin. If $c > 0$, the origin is on the positive side of the line; otherwise it is on the negative side of the line. If the line goes through the origin, a point on one of the axes can be used.

THEOREM 8–9.* *The plane* $ax + by + cy + d = 0$ *separates xyz-space into two regions, called half-spaces, such that* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$ *lie in different half-spaces if and only if the numbers* $(ax_1 + by_1 + cz_1 + d)$ *and* $(ax_2 + by_2 + cz_2 + d)$ *have opposite signs.*

The set of points that satisfy two linear inequalities in x and y simultaneously is the set common to two half-planes. This set is also convex, for if the segment $\overline{P_1P_2}$ lies in each of two convex sets, it lies in their intersection.

The set of points that satisfy three linear inequalities,

$$a_1x + a_2y + a_3 > 0$$
$$b_1x + b_2y + b_3 > 0$$
$$c_1x + c_2y + c_3 > 0,$$

is the set common to three half-plane and it is also convex. It may be a triangular region like that shaded in Fig. 8–10 or it may be a region bounded by a segment and two rays like region 2 in Fig. 8–10. The solution set may be empty; the third boundary line may not enter the convex region defined by the first two inequalities. For example, the inequalities

$$x > 0, \qquad y > 0, \qquad x + y + 1 < 0 \qquad \text{or} \qquad -x - y - 1 > 0$$

have no common solution, since the line $x + y + 1 = 0$ does not enter the first quadrant corresponding to $x > 0$, $y > 0$. Special cases arise if the boundary lines are concurrent or parallel.

The set of points which satisfy four linear inequalities may be a convex quadrangular set like that bounded by $BCDE$ in Fig. 8–10 or it may be the empty set.

EXAMPLE 8–17. Determine graphically the region in the xy-plane, as the intersection of half-planes, which is defined by the inequalities

(a) $4x + 2y > 8, \qquad 2x + 4y > 8, \qquad x > 0, \qquad y > 0.$

The origin lies on the negative side of the lines $4x + 2y - 8 = 0$ and $2x + 4y - 8 = 0$. These two lines intersect at the point $E(\frac{4}{3}, \frac{4}{3})$. The required region is that part of the first quadrant whose boundary is the ray $\overrightarrow{CX}$, the segments $\overline{CE}$ and $\overline{EB}$, and the ray $\overrightarrow{BY}$ as shown in Fig. 8–11.

In a similar manner, it could be shown that the region corresponding to

(b) $4x + 2y < 8, \qquad 2x + 4y < 8. \qquad x > 0, \qquad y > 0$

---

* A space separation postulate is needed and the proof of this theorem is then like that of Theorem 8–8. The details have been omitted.

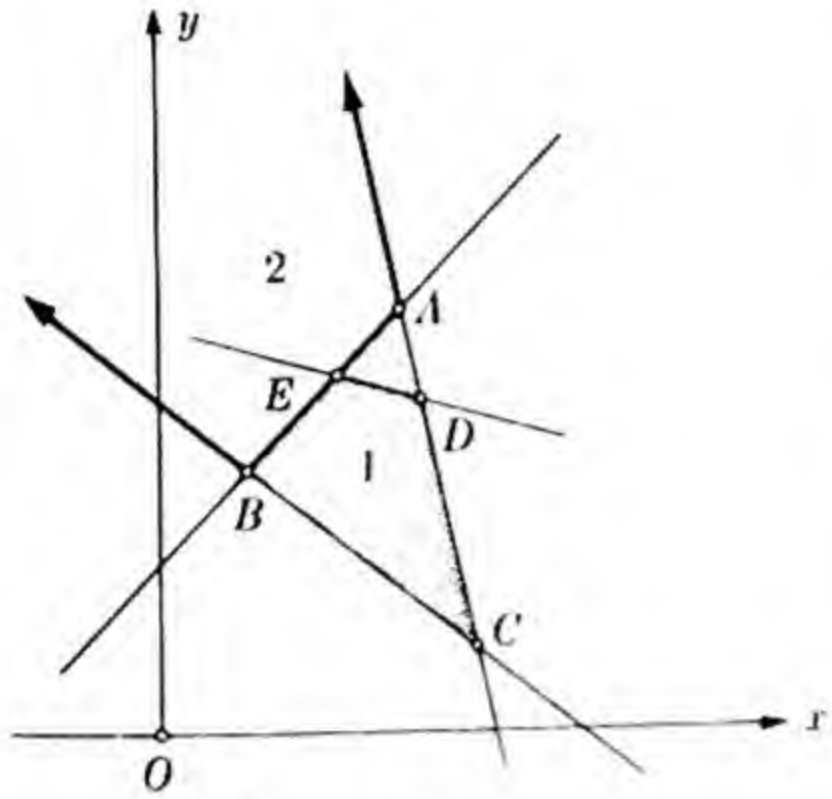


FIGURE 8-10

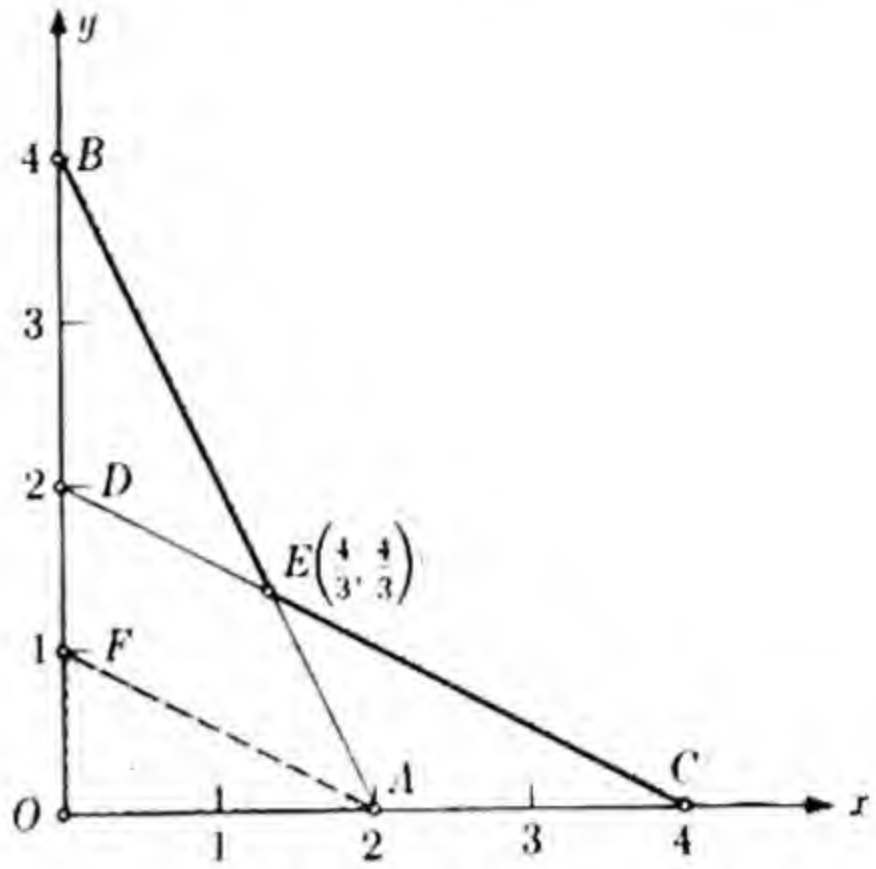


FIGURE 8-11

is the interior of the convex quadrilateral $OAED$. On the other hand, the inequalities

(c) $4x + 2y > 8, \quad 2x + 4y < 4, \quad x > 0, \quad y > 0$

have no common solution.

A similar analysis applies to linear inequalities in x, y, and z. The corner points of the corresponding convex polyhedron can be found by solving all possible sets of three equations which correspond to the boundary planes. The drawing is now more difficult to make. After any corner point is found, its coordinates are tested to see if they satisfy the residual inequalities. For example, in Fig. 8-11, it is noted that $E(\frac{4}{3}, \frac{4}{3})$ does not satisfy $2x + 4y < 4$.

Any inequality can be changed to an equality by introducing an auxiliary variable (called a *slack variable*) which is positive (Section 7-6, Theorem 7-18). For example, the inequality $2x - 4y \leqq 8$ is equivalent to

$$2x - 4y + u = 8, u \geqq 0,$$

where u is the slack variable.

EXAMPLE 8-18. Solve the sets of inequalities of Example 8-17 for x and y by introducing two slack variables, u and v. Illustrate geometrically in the uv-plane.

(a) $4x + 2y > 8, \quad 2x + 4y > 8, \quad x > 0, \quad y > 0.$

The inequalities are equivalent to:

$$4x + 2y = 8 + u, \quad 2x + 4y = 8 + v, \quad (u, v, x, y > 0).$$

The first two equations may be solved to find

$$6x = (8 + 2u - v), \qquad 6y = 8 - u + 2v,$$

where u and v may be any positive values that make x and y both positive. This is illustrated in the uv-plane by drawing the lines corresponding to $x = 0$ and $y = 0$ and determining in the first quadrant where x and y are positive. The origin makes both $8 + 2u - v$ and $8 - u + 2v$ positive, so the region is as marked in Fig. 8–12.

(b) The system $4x + 2y < 8$, $2x + 4y < 8$, $x > 0$, $y > 0$ can be solved in a similar manner and the uv-diagram drawn.

(c) The system $4x + 2y > 8$, $2x + 4y < 4$, $x > 0$, $y > 0$ is equivalent to

$$\begin{array}{l|l} 4x + 2y = 8 + u & -1 \\ 2x + 4y = 4 - v & 2 \end{array} \quad (x, y, u, v > 0).$$

If x is eliminated from these equations,

$$6y = -u - 2v,$$

which cannot be positive for u and v positive, and hence the given set of inequalities has no solution.

Maximum and minimum of a linear function. If the convex region defined by a set of linear inequalities is unbounded, as in Example 8–17(a), a given linear function $ax + by$ (or $ax + by + k$) may have no maximum value (or minimum value); that is to say, the absolute value of $ax + by$ can be made arbitrarily large. If, however, the region is such that x and y are bounded above, then the set of values of $ax + by$ is also bounded above and has a maximum value (Least Upper Bound Axiom, Section 7–9).

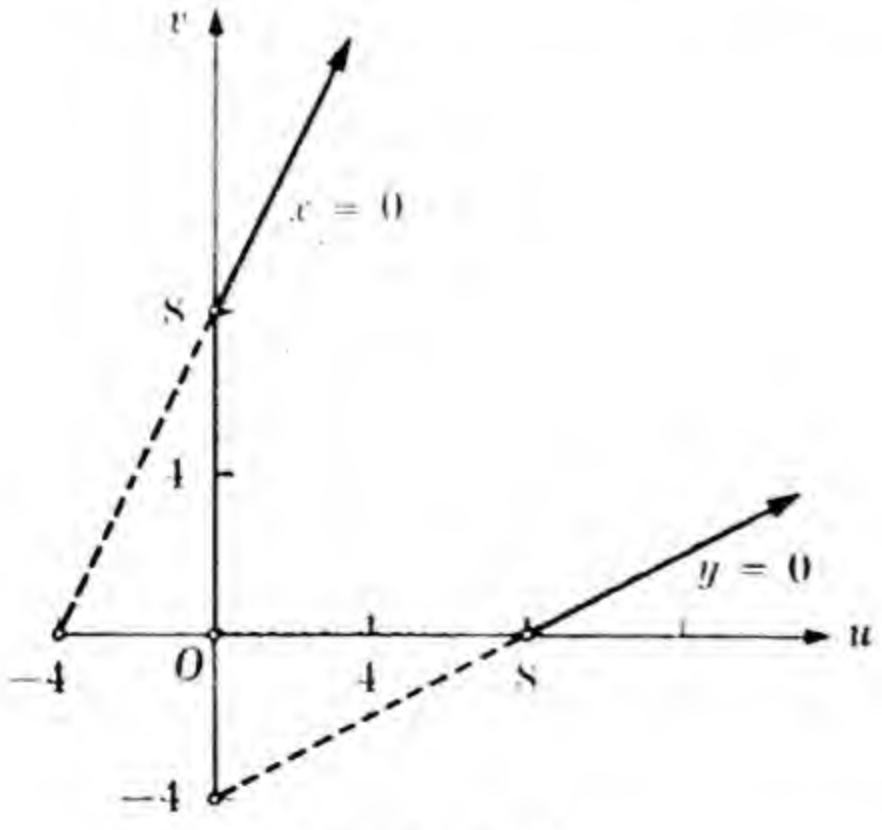


FIGURE 8–12

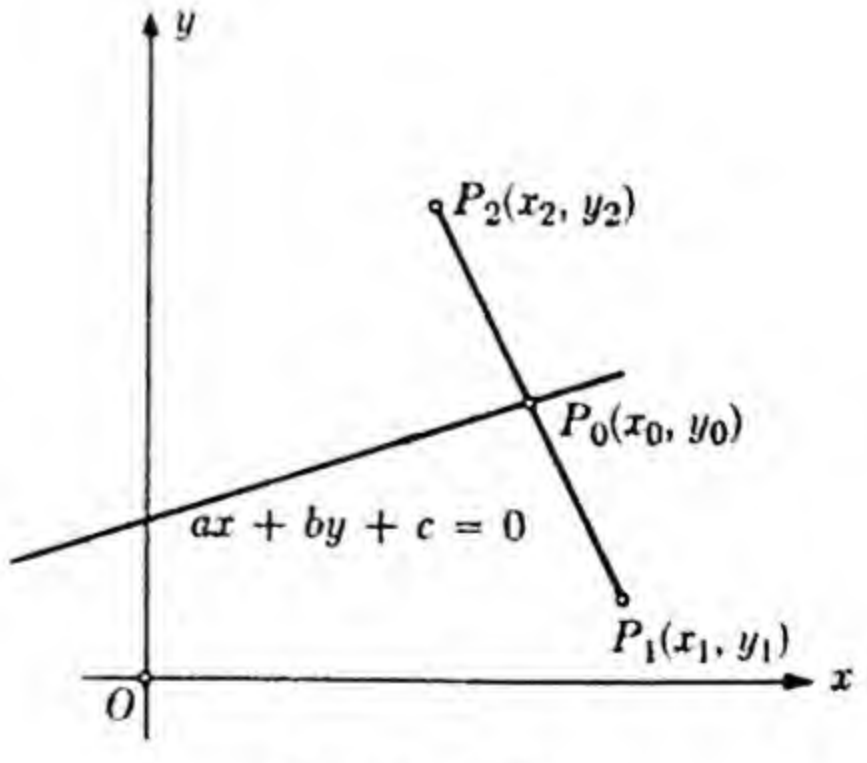


FIGURE 8–13

Similarly, if the region is such that x and y are bounded below, $ax + by$ has a minimum value. The following theorem gives a method for finding the maximum and minimum of $ax + by$, called the *objective function*, over the convex set defined by a set of linear inequalities.

THEOREM 8-10. *A linear objective function, $ax + by + k$, evaluated over a convex region defined by a set of linear inequalities takes on its maximum and minimum values at corner points.*

The theorem is first proved for the special case where the region is a line segment (Fig. 8-13):

Suppose $P_0(x_0, y_0)$ is any point *between* P_1 and P_2, the endpoints of the line segment. Suppose the value of $ax + by$ at P_0 is $-c$, so that $ax_0 + by_0 + c = 0$ and hence P_0 is on the line $ax + by + c = 0$. Then the values of $ax + by$ at P_1 and P_2 are such that $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$ have opposite signs (Theorem 8-8). Hence $(ax_0 + by_0 + c) = 0$ is between $(ax_1 + by_1 + c)$ and $(ax_2 + by_2 + c)$. Adding k and subtracting c does not affect the results, so that $(ax_0 + by_0 + k)$ is between $(ax_1 + by_1 + k)$ and $(ax_2 + by_2 + k)$, and the greatest and least values of the objective function must occur at the endpoints of the segment.

Remark 1. The proof implied that $ax + by + c = 0$ is not the line P_1P_2. If these lines coincide, then $ax + by$ takes on the same value $(-c)$ at every point of the segment P_1P_2.

Now suppose that the region is a convex polygon and that Q is a point in the interior of the polygon. Then Q lies between two vertices, say A and B, or between the vertex A and a point P between two vertices B and C (Fig. 8-14). In the first case, the value of $ax + by + k$ at Q is between its values at A and B and in the second case the value at Q is between the values at A and P, where the value at P is between the values at B and C. It follows that the maximum or minimum must occur at a corner point.

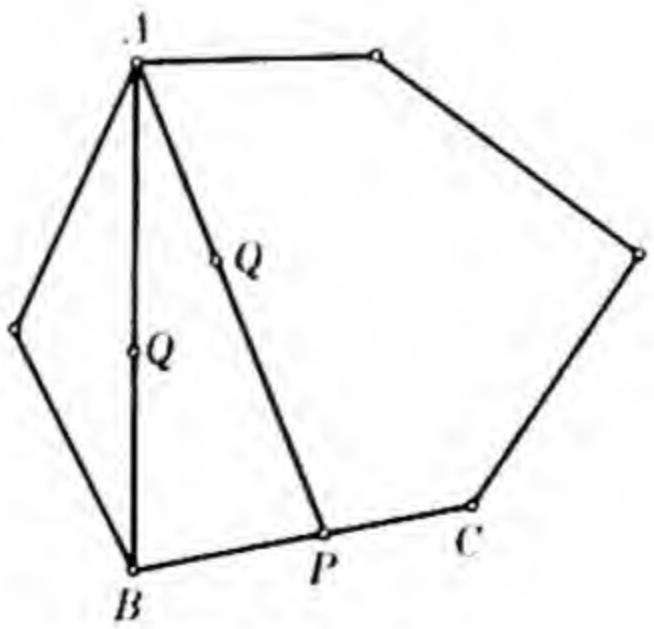


FIGURE 8-14

Remark 2. If the region is unbounded, $ax + by$ may still have either a maximum or minimum at a corner point, depending upon whether a and b are positive or negative and also upon the nature of the unbounded region.

Remark 3. If the convex region is in three spaces, an analogous theorem applies to the objective function $ax + by + cz + k$.

EXAMPLE 8–19. Find the maximum and minimum value of $3x + 4y$ over the convex region defined by

(a) $4x + 2y \geqq 8$, $2x + 4y \geqq 8$, $x \geqq 0$, $y \geqq 0$;

(b) $4x + 2y \leqq 8$, $2x + 4y \leqq 8$, $x \geqq 0$, $y \geqq 0$.

(See Example 8–17 and Fig. 8–11.)

For part (a), the region is unbounded and $3x + 4y$ can be made arbitrarily large. The finite corners of the region are $C(4, 0)$, $E(\frac{4}{3}, \frac{4}{3})$, and $B(0, 4)$. The values of $3x + 4y$ at the points are $V_C = 12$, $V_E = \frac{28}{3}$, $V_B = 16$. Hence the minimum of $3x + 4y$ over the convex region occurs at E and has the value $\frac{28}{3}$.

For part (b), the region is the interior of the convex quadrilateral $OAED$, where $O(0, 0)$, $A(2, 0)$, $E(\frac{4}{3}, \frac{4}{3})$, $D(0, 2)$. The values are $V_O = 0$, $V_A = 6$, $V_E = \frac{28}{3}$, $V_D = 8$. The minimum value is 0 at O and the maximum value is $\frac{28}{3}$ at E.

EXAMPLE 8–20. Find the maximum and minimum values of

$$V = 2x + 3y - z$$

subject to

$x \geqq 0$, $y \geqq 0$, $z \geqq 0$, $x + 2y + z \leqq 4$, $2x + y - z \geqq 0$.

The convex region is a pyramid which has O as its vertex and the quadrilateral $ABDE$ as its base. $A(4, 0, 0)$, $B(0, 2, 0)$, $D(0, \frac{4}{3}, \frac{4}{3})$,

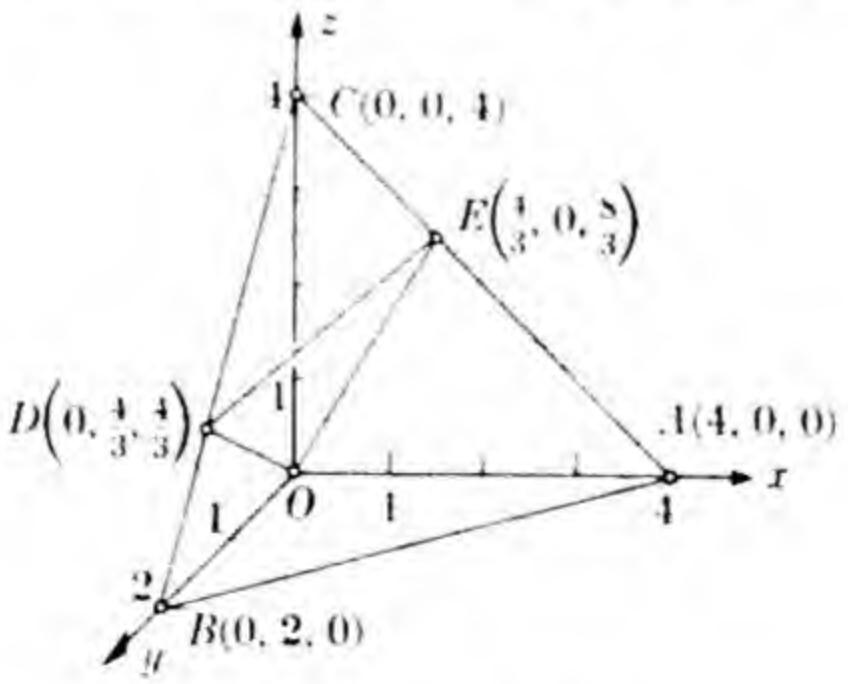


FIGURE 8–15

$E(\frac{4}{3}, 0, \frac{8}{3})$ are found by solving the corresponding planes, three at a time. Drawing a figure reduces the amount of work to be done. A number of the corner points appear directly, and it is not necessary to solve all ten sets of such equations. The point $D(0, \frac{4}{3}, \frac{4}{3})$ is found by solving $x = 0$, $2x + y - z = 0$, and $x + 2y + z = 4$; the point $E(\frac{4}{3}, 0, \frac{8}{3})$, from $y = 0$, $2x + y - z = 0$, and $x + 2y + z = 4$. Note that the point $x = 0$, $y = 0$, $x + 2y + z = 4$, or $C(0, 0, 4)$ does not satisfy $2x + y - z \geqq 0$, whereas A and B do.

$$V_O = 0, \qquad V_A = 8, \qquad V_B = 6, \qquad V_D = \tfrac{8}{3}, \qquad V_E = 0.$$

The maximum value of V is $V_A = 8$. The minimum is 0, which occurs not only at O and E but at every point on the segment $\overline{OE}$ where $y = 0$ and $z = 2x$.

Problem Set 8-7

1. Sketch the lines $x - y + 2 = 0$ and $5x - y - 10 = 0$ and mark the four regions $(+\,+)$, $(+\,-)$, $(-\,+)$ and $(-\,-)$ according to the sign of $x - y + 2$ and $5x - y - 10$.

2. (a) Three nonconcurrent and nonparallel lines separate the plane into seven regions. Let the equations of the lines be

$$a_1x + a_2y + a_3 = 0; \qquad b_1x + b_2y + b_3 = 0, \qquad c_1x + c_2y + c_3 = 0,$$

selected so that for points inside the triangle formed by them the three left-hand members are all positive. Mark the other regions by a combination of $+$ and $-$ signs. For which set of signs are there no points?

(b) If the equations had been selected so that the triangular region is marked $+\,-\,+$, for which set of signs is there no region?

3. Determine graphically the region in the xy-plane which is defined by the following inequalities. Determine the corner points of the required region.

(a) $x \geqq 0$, $y \geqq 0$, $3x + 2y - 6 \leqq 0$
(b) $x \geqq 0$, $y \geqq 0$, $2x + 4y - 5 \geqq 0$
(c) $x \geqq 0$, $y \geqq 0$, $3x + 2y + 6 \leqq 0$
(d) $x \leqq 2$, $y \leqq 3$, $3x + 2y \geqq 0$

4. Proceed as in problem 3 for

(a) $2x + 3y \leqq 6$, $-x + y \leqq 2$, $x + 3y \leqq 3$
(b) $2x + 3y \geqq 6$, $-x + y \leqq 2$, $x + y \leqq 3$

5. Proceed as in problem 3 for

(a) $-x + y + 1 \geqq 0$, $2x + 3y - 6 \geqq 0$, $x - 2y + 4 \geqq 0$
(b) $x - y + 2 \geqq 0$, $-2x - y + 11 \geqq 0$, $x + 2y - 7 \geqq 0$

6. Proceed as in problem 3 for

(a) $2x + 3y \geqq 6, \quad 3x + 2y \geqq 6, \quad x \geqq 0, \quad y \geqq 0$

(b) $4x - 2y \leqq 8, \quad 2x + 4y \leqq 8, \quad x \geqq 0, \quad y \geqq 0$

(c) $2x - y + 3 \leqq 0, \quad x + 2y - 4 \leqq 0, \quad x \geqq 0, \quad y \geqq 0$

7. Solve each of the sets of inequalities (a), (b), (c) in problem 6 for x and y in terms of two slack variables u and v, and illustrate geometrically in the uv-plane. (See Fig. 8–12.)

8. Find the maximum and minimum values of the linear objective function $7x + 5y - 3$ over the convex region defined by the inequalities in the following problems. Use the region and the corner points already found.

(a) $x \geqq 0, \quad y \geqq 0, \quad 3x + 2y \leqq 6$ Problem 3(a)

(b) $x \geqq 0, \quad y \geqq 0, \quad 2x + 4y - 5 \geqq 0$ Problem 3(b)

(c) $2x + 3y \leqq 6, \quad -x + y \leqq 2, \quad x + 3y \leqq 3$ Problem 4(a)

(d) $2x + 3y \geqq 6, \quad -x + y \leqq 2, \quad x + 3y \leqq 3$ Problem 4(b)

(e) $-x + y + 1 \geqq 0, \quad 2x + 3y - 6 \geqq 0, \quad x - 2y + 4 \geqq 0$ Problem 5(a)

(f) $x - y + 2 \geqq 0, \quad -2x - y + 11 \geqq 0, \quad x + 2y - 7 \geqq 0$ Problem 5(b)

9. Find the maximum and minimum values of the linear objective function $3x + y + 2$ over the convex region defined by the inequalities

$$2x + y + 9 \geqq 0, \quad -x + 3y + 6 \geqq 0, \quad x + 2y - 3 \leqq 0, \quad x + y \leqq 0.$$

10. Determine graphically the region in space which is defined by the following inequalities. Determine the corner points of the required region.

(a) $x \geqq 0, \quad y \geqq 0, \quad z \geqq 0, \quad x + y + z \leqq 4, \quad x + y - z \leqq 0$

(b) $x \geqq 0, \quad y \geqq 0, \quad z \geqq 0, \quad x + y + z \leqq 4, \quad 3x + y - 3z \geqq 0$

(c) $x \leqq 2, \quad y \leqq 3, \quad z \leqq 4, \quad 6x + 4y + 3z \geqq 24$

11. Determine the maximum and minimum values of the objective function $x - 2y + 3z$ over each of the regions in problems 10(a), (b), (c). Use the corner points already found.

8–9 Linear programming. Economic problems which are concerned with minimizing or maximizing a linear objective function subject to a set of linear inequalities are referred to as *linear programming* problems. The objective function may be total cost, profit, or a similar economic quantity. The constraints are usually due to limitations on material resources, such as materials, labor, capital, or equipment. Simplified versions of such problems are given as examples. One method of solution of such a problem is the "corner method" based on Theorem 8–10. A second method is the "method of successive elimination," in which the inequalities are replaced by equalities through the introduction of slack variables and the objective function is expressed in terms of these slack

variables. In such economic problems, all the original and the slack variables are positive or zero. Both geometric and algebraic procedures may play a role in the solution of such problems.

The diet problem. In this problem a number of foods in quantities $x, y, z, \ldots$ are purchased. It is known that each food contains given amounts of nutritional elements that are specified by a set of constants, which may be given in matrix form. A set of minimum requirements for each nutrient is specified. The cost of each unit of food is given and the problem is to minimize the total cost subject to the constraints placed by the nutritional requirements.

EXAMPLE 8–21. A well-known nursery rhyme goes: "Jack Sprat could eat no fat, his wife could eat no lean, and so you see between the two they licked the platter clean." (a) Suppose the Sprats buy only beef at \$1 a pound which contains 90% lean meat and 10% fat, and pork at \$.50 per pound which contains 70% lean meat and 30% fat. Jack's weekly requirement is at least 6 lb of lean meat and his wife's requirement is at least 2 lb of fat. How much beef and pork should they buy each week to minimize the cost?

(b) Solve the problem if pork costs \$.85 per pound.

(a) Let x and y be the number of pounds of beef and pork purchased. Jack's requirement is expressed as: $0.9x + 0.7y \geqq 6$, his wife's as $0.1x + 0.3y \geqq 2$, and the total cost in dollars is

$$Q = x + \tfrac{1}{2}y.$$

The problem is to minimize Q subject to the constraints

$$\begin{aligned} 9x + 7y &\geqq 60 \\ x + 3y &\geqq 20, \qquad \text{where} \quad x \geqq 0, \qquad y \geqq 0. \end{aligned}$$

First solution. If the lines $9x + 7y = 60$, $x + 3y = 20$ are drawn and the equations solved simultaneously, it is noted that the convex region which corresponds to the inequalities has three finite corner points $A(20, 0)$, $B(0, 60/7)$, $C(2, 6)$. If the cost function $Q = x + \frac{1}{2}y$ is evaluated at these points, it is found that $Q_A = 20$, $Q_B = 30/7$, $Q_C = 5$. Consequently, the cost is least if the diet consists of all pork. This is consistent with the geometric fact that the family of lines (Fig. 8–16) $x + \frac{1}{2}y = Q$ will first give a line through B as Q increases.

(b) If pork costs \$.85 a pound, the cost is now $Q = x + 0.85y$, so that

$$Q_A = \$20, \quad Q_B = \$.85(60/7) = \$7.29, \quad Q_C = \$2 + \$.85(6) = \$7.10.$$

Now the cost is least on a weekly diet of 2 lb of beef and 6 lb of pork.

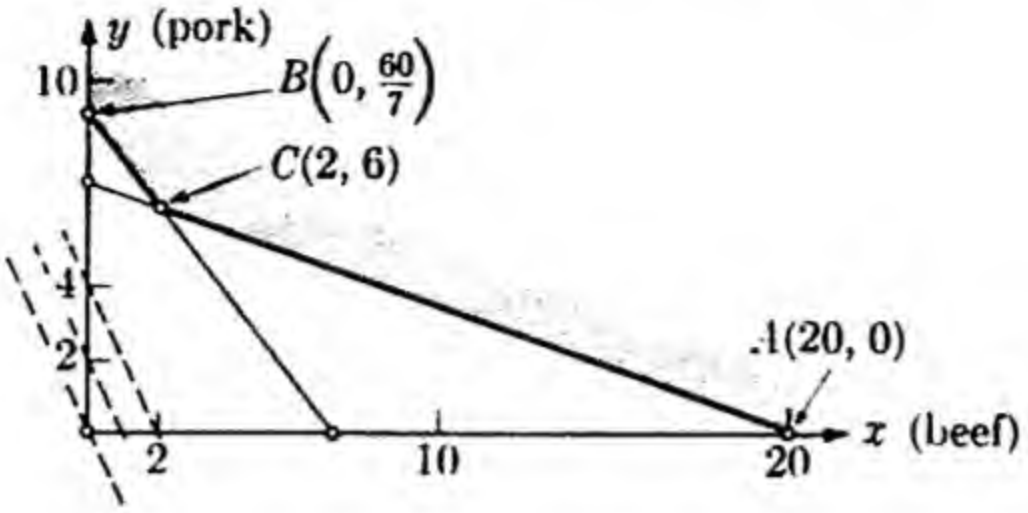


FIGURE 8–16

As Q increases, the family of lines $x + 0.85y = Q$ passes through C first (not shown).

Second solution. The inequalities may be replaced by the equalities

$$9x + 7y = 60 + u,$$
$$x + 3y = 20 + v,$$

where x, y, u, v are all $\geqq 0$. The solution of these equations (the details are left as an exercise) gives

$$x = 2 + \tfrac{3}{20}u - \tfrac{7}{10}v,$$
$$y = 6 - \tfrac{1}{20}u + \tfrac{9}{20}v.$$

(b) If $Q = x + 0.85y$, then

$$20Q = 142 + 2.15u + 0.65v,$$

and the minimum cost occurs for $u = v = 0$ and is \$7.10 and $x = 2$ and $y = 6$.

(a) If the cost is $Q = x + \frac{1}{2}y$, then

$$Q = 5 + \tfrac{1}{8}u - \tfrac{1}{8}v.$$

The cost is minimized by making $u = 0$ and v as large as possible. Reference to the value of x, which must be positive, yields

$$0 \leqq x = 2 - \tfrac{7}{10}v \qquad \text{or} \qquad v \leqq \tfrac{40}{7}.$$

Using the value $v = \frac{40}{7}$ gives the least value of Q,

$$Q_{\min} = 5 - \tfrac{1}{8} \cdot \tfrac{40}{7} = 4\tfrac{2}{7}.$$

which agrees with the previous results.

EXAMPLE 8–22. Four foods are purchased in amounts x_1, x_2, x_3, x_4 and their calorie content, vitamin content, and prices per unit quantity

are given by the following matrix

$$\begin{array}{l} \\ \text{Calories} \\ \text{Vitamins} \\ \text{Prices} \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 2 & 0 & 1 & 3 \\ 0 & 2 & 1 & 2 \\ 4 & 10 & 12 & 15 \end{pmatrix} \end{array}.$$

The minimum calorie requirement is 15 units and the minimum vitamin requirement is 8. How much of each food should be purchased to satisfy these requirements, and to minimize the total cost?

The problem is to determine x_1, x_2, x_3, x_4 to minimize

$$Q = 4x_1 + 10x_2 + 12x_3 + 15x_4$$

subject to the inequalities

$$2x_1 + x_3 + 3x_4 \geqq 15, \qquad 2x_2 + x_3 + 2x_4 \geqq 8,$$
$$x_1 \geqq 0, \qquad x_2 \geqq 0, \qquad x_3 \geqq 0, \qquad x_4 \geqq 0.$$

First solution. (Corner method.) To find all the corners, it is necessary to select four out of six equations in all possible ways. Although there would be 15 such sets, many of these can be eliminated quickly. At least two of the variables must be zero and when three or four of these are zero, the constraints cannot be satisfied so as to minimize the cost. There remain six important corner points to consider:

$$A(0, 0, x_3, x_4), \qquad B(0, x_2, 0, x_4), \qquad C(0, x_2, x_3, 0),$$
$$D(x_1, 0, 0, x_4), \qquad E(x_1, 0, x_3, 0), \qquad F(x_1, x_2, 0, 0).$$

In the first set, the equations to determine x_3, x_4 are

$$x_3 + 3x_4 = 15, \qquad x_3 + 2x_4 = 8$$

with solutions $x_4 = 7$, $x_3 = -6$, and this does not satisfy the constraint $x_3 \geqq 0$. Similarly, the other corner points can be found as $A(0, 0, -6, 7)$, $B(0, -1, 0, 5)$, $C(0, -1, 10, 0)$, $D(\frac{3}{2}, 0, 0, 4)$, $E(\frac{7}{2}, 0, 8, 0)$, $F(\frac{15}{2}, 4, 0, 0)$. The first three are rejected and the value of Q is computed for the other three:

$$Q_D = 4 \cdot \tfrac{3}{2} + 15 \cdot 4 = 66; \qquad Q_E = 4 \cdot \tfrac{7}{2} + 12 \cdot 8 > 66;$$
$$Q_F = 4 \cdot \tfrac{15}{2} + 40 = 70.$$

Hence the best buy is $\frac{3}{2}$ units of the first food and 4 units of the fourth food. In this example, the minimum requirements are exactly satisfied.

Second solution. (Elimination method.) Before giving the second solution, the details may be simplified by noting that under the given condi-

tions, none of the third food would be purchased. The data are such that for 60 monetary units all spent on the third food, 5 calories and 5 vitamin units are available; while the same sum spent on the fourth food would yield 12 calories and 8 vitamin units. A similar argument does not, however, apply to the second food, for which 12 vitamin units could be purchased, but no calories. Let the remaining variables now be called x, y, z, so that the problem becomes: minimize

$$Q = 4x + 10y + 15z,$$

subject to the inequalities

$$2x + 3z \geqq 15, \qquad 2y + 2z \geqq 8, \qquad x, y, z \geqq 0.$$

The inequalities are replaced by the equations

$$2x + 3z = 15 + u$$
$$y + z = 4 + v,$$

$x, y, z, u, v \geqq 0$. These equations are solved for x and z in terms of the other variables.

$$\begin{aligned} 2x - 3y &= 3 + u - 3v \\ x &= \tfrac{1}{2}(3 + 3y + u - 3v) \\ z &= 4 - y + v. \\ Q &= (6 + 6y + 2u - 6v) + 10y + 60 - 15y + 15v \\ &= 66 + y + 2u + 9v. \end{aligned}$$

Hence the minimum value of Q is 66, which occurs when $y = u = v = 0$, $x = \frac{3}{2}$ and $z = 4$, as before.

The method depends upon the fact that the coefficients of y, u, v are positive. If other variables had been eliminated, the solution could not be found so easily. In the general case, attempts are made to express Q in terms of u, v and one of x, y, z.

Activity analysis. If several articles are to be produced in quantities x, y, z, . . . , and if each article requires specified amounts of material, labor, and equipment and, finally, if the articles can be sold at profits which are known, then the linear programming problem is to maximize the profit subject to the constraints that are imposed because the resources available are limited, and the further conditions that all variables are positive or zero.

EXAMPLE 8–23. A poultry raiser plans to raise chickens, ducks, and turkeys. He has room for only 200 birds and wishes to limit the number

of turkeys to a maximum of 25, the number of turkeys and ducks to a maximum of 100. His estimated profits are \$1, \$2, and \$3 on each chicken, duck and turkey, respectively. How many of each should he raise to maximize his profit?

Solution. Let the number of chickens, ducks, and turkeys be x, y, z, respectively. The problem is to maximize

$$P = x + 2y + 3z$$

subject to the inequalities

$$x + y + z \leqq 200, \qquad y + z \leqq 100, \qquad z \leqq 25, \qquad x, y, z \geqq 0.$$

There are eight corner points in the first octant formed by the three planes $x + y + z = 200$, $y + z = 100$, $z = 25$. These are the vertices of the convex region defined by the inequalities. It is left as an exercise to verify the following table:

	x	y	z	P	
O	0	0	0	0	
A	200	0	0	200	
B	100	100	0	300	
C	0	100	0	200	
D	0	0	25	75	
E	175	0	25	250	
F	100	75	25	325	max.
G	0	75	25	225	

The algebraic solution is simple:

$$\begin{aligned} x + y + z + u &= 200 \\ y + z + v &= 100 \\ z + w &= 25, \end{aligned}$$

$$(x, y, z, u, v, w \geqq 0).$$

These equations are solved for x, y, z in terms of u, v, w:

$$\begin{aligned} x &= 100 - u + v, \\ y &= 75 - v + w, \\ z &= 25 - w, \end{aligned}$$

so that

$$\begin{aligned} P &= x + 2y + 3z \\ &= 325 - u - v - w. \end{aligned}$$

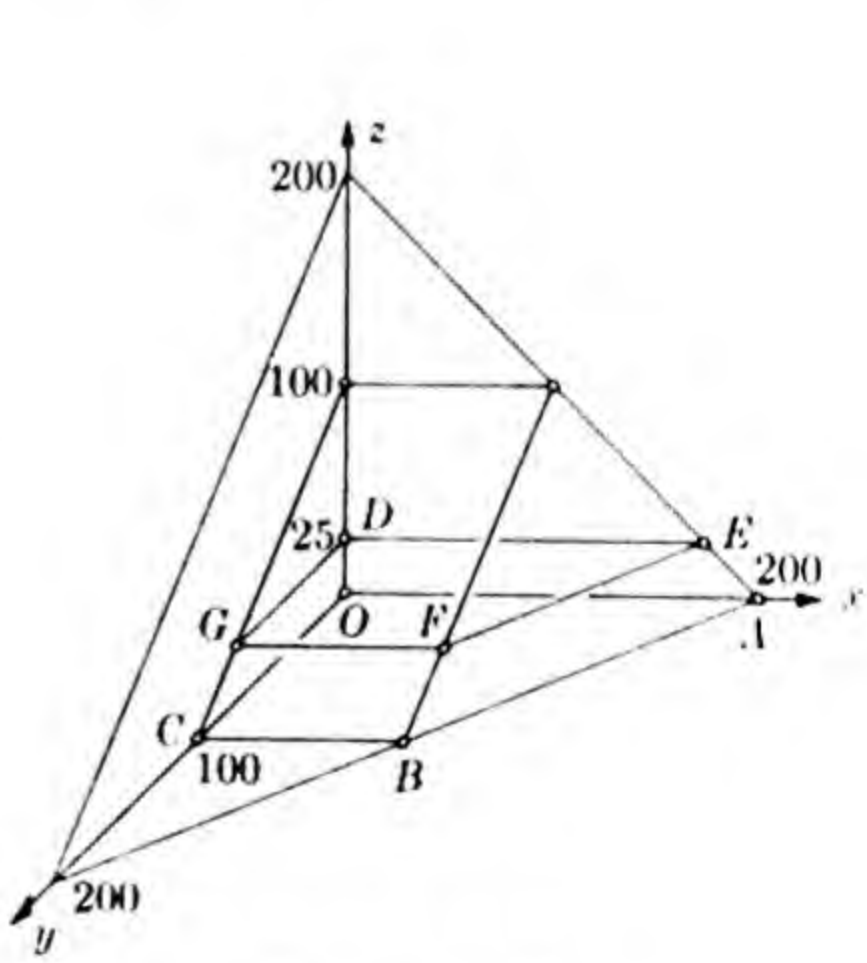


FIGURE 8-17

The maximum profit, $P = 325$, occurs when $u = v = w = 0$, and $x = 100$, $y = 75$, $z = 25$.

The situation would be different if the profit on each turkey were less than \$2. In that case, it would be found* that no turkeys, 100 chickens, and 100 ducks should be raised.

EXAMPLE 8–24. A manufacturer wishes to produce two commodities in quantities x and y. Each commodity requires material, labor, and equipment use, in amounts given by the following table. The number of units available is also specified:

	x	y	*Available*
Material	5	4	100
Labor	3	8	172
Equipment	3	1	53

Find the maximum profit, if the profit function is

$$\text{(a) } P_1 = x + 2y, \qquad \text{(b) } P_2 = 2x + y.$$

The problem is to maximize the profit, subject to the linear inequalities

$$\begin{aligned} 5x + 4y &\leqq 100, \\ 3x + 8y &\leqq 172, \\ 3x + \ y &\leqq \ 53, \qquad x, y \geqq 0. \end{aligned}$$

The convex region defined by the inequalities is a pentagon $OABCD$ as shown in Fig. 8–18. The coordinates of the points and the values of P_1 and P_2 are summarized in the following table.

	(x, y)	P_1	P_2
O	(0, 0)	0	0
A	$(\frac{53}{3}, 0)$	$\frac{53}{3}$	$\frac{106}{3}$
B	(16, 5)	22	37
C	(4, 20)	44	28
D	$(0, \frac{43}{2})$	43	$\frac{43}{2}$

P_1 is a maximum of 44 when $x = 4$, $y = 20$. This is consistent with the geometric fact that the family of lines $x + 2y = P$ would have its greatest value when passing through the corner point C. The profit P_2 is a maximum of 37 when $x = 16$, $y = 5$, and a geometric interpretation related to the family of lines $2x + y = P$ can be given.

---

* Left as an exercise.

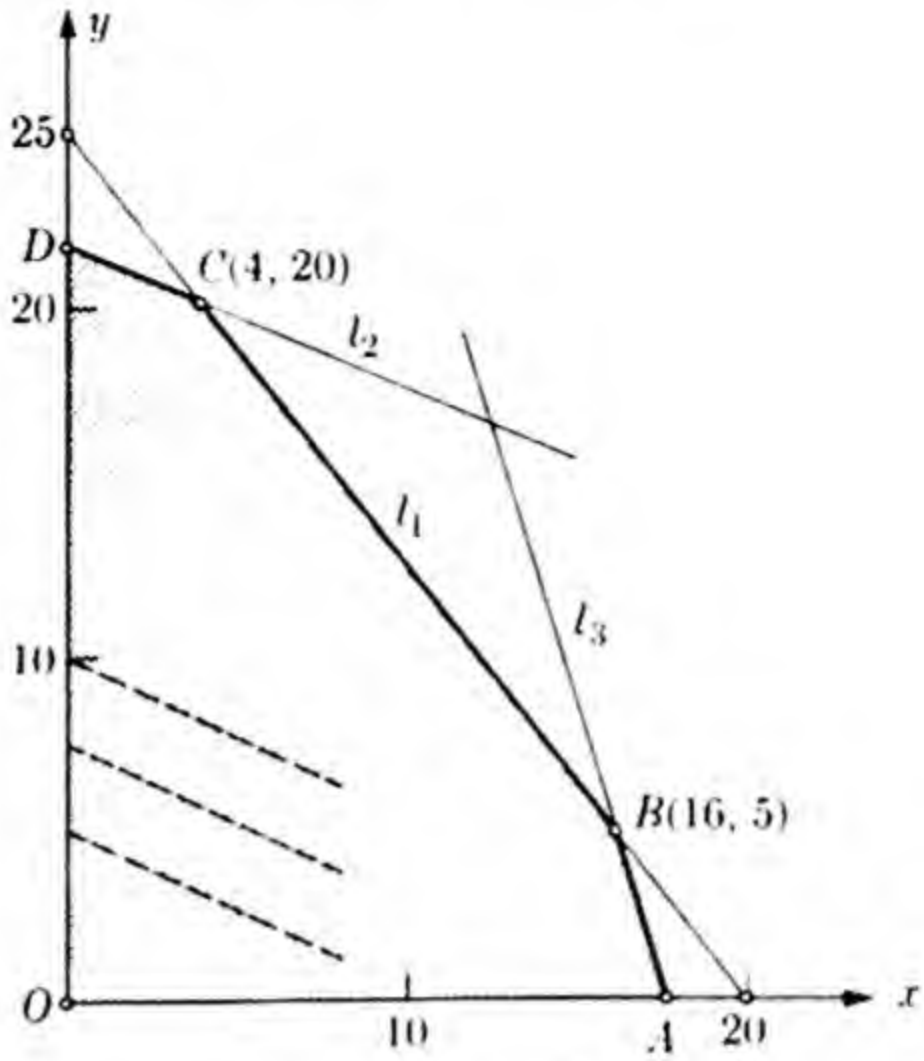


FIGURE 8-18

Second solution. The given inequalities are equivalent to the equations

$$5x + 4y = 100 - u,$$
$$3x + 8y = 172 - v,$$
$$3x + \ y = 53 - w,$$

where $x, y, z, u, v, w \geqq 0$. If the first two of these equations are solved for x and y, it is found

$$28x = 112 - 8u + 4v \,\big|\, 1$$
$$28y = 560 + 3u - 5v \,\big|\, 2$$
$$28P_1 = 1232 - 2u - 6v.$$

P_1 has its maximum value, $P_1 = 44$, for $u = v = 0$. If the first and third of these equations are solved for x and y, it is found that

$$7x = 112 + u - 4w \,\big|\, 2$$
$$7y = 35 - 3u + 5w \,\big|\, 1$$
$$7P_2 = 259 - u - 3w,$$

so that P_2 has its maximum value $P_2 = 37$ for $u = w = 0$. The method depends upon the fact that the coefficients of u, v, w are all negative. If some are positive, the method fails, but may be combined with the corner method to find the solution. For example,

$$7P_1 = 182 - 5u + 6w.$$

The maximum value of P_1 will occur for $u = 0$, which corresponds to $5x + 4y = 100$. Furthermore, this occurs at the ends of the segment of this line which is part of the boundary of the convex region. These are the points B and C, and it is readily verified P_1 is greater at C than at B.

The general linear programming problem can be stated as follows: Maximize (or minimize) the linear objective function

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the inequalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leqq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leqq b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &, \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leqq b_n. \\ x_1, x_2, \ldots, x_n &\geqq 0. \end{aligned}$$

The inequalities can be replaced by equalities by the introduction of m slack variables $u_1, u_2, \ldots, u_m \geqq 0$. This gives m equations in $(n + m)$ variables to be satisfied by the zero or positive values of the variables. In compact matrix form, the problem can be written: Maximize (or minimize) the objective function $\mathbf{CX}$, subject to the matrix equation

$$\mathbf{AX}_1 = \mathbf{B},$$

where $\mathbf{C}$ is the row vector $(c_1\, c_2\, c_3 \ldots c_n)$, $\mathbf{X}$ is the column vector $(x_1\, x_2\, x_3 \ldots x_n)^T$, $\mathbf{A}$ is the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

of m rows and $n + m$ columns, $\mathbf{X}_1$ is the column vector

$$(x_1\, x_2 \ldots x_n\, u_1\, u_2 \ldots u_m)^T$$

and $\mathbf{B}$ is the column vector $(b_1\, b_2 \ldots b_m)^T$. In practical problems the very large number of conditions to be met makes the calculations very difficult, and special methods adapted to high-speed computing machines have been devised. The interested reader will find a rapidly growing literature related to the subject available.*

---

* See, for example, Dorfman, Samuelson, and Solow, *Linear Programming and Economic Analysis*, McGraw-Hill, 1958; Saul J. Gass, *Linear Programming*, McGraw-Hill, 1958. This latter book contains an excellent bibliography.

Problem Set 8–8

1. (a) How much beef at \$.90 a lb which contains 90% lean meat and 10% fat, and how much pork at \$.60 a lb which contains 75% lean meat and 25% fat, should be purchased to provide at least 8 lb of lean meat and 2 lb of fat while keeping the cost as small as possible.

(b) Solve the corresponding problem if the cost of pork is changed to \$.75 a lb.

Solve the problem by the corner method and also by the elimination method (see Example 8–21).

2. Three foods are purchased in amounts x, y, z. Their calorie and vitamin contents, their prices and minimum nutritional requirements are as follows:

	x	y	z	*Requirement*
Calories	3	1	4	25
Vitamins	0	3	3	12
Price	6	8	12	

How much of each food should be purchased to satisfy these requirements and to minimize the total cost?

Solve the problem by the corner method and by the elimination method.

3. (a) Verify the table given in the solution of Example 8–23, and (b) justify the statement that if the profit on each turkey is less than \$2, no turkeys should be raised.

4. A poultry farmer decides to raise 200 fowl consisting of chickens, ducks (not to exceed 100 in number), and turkeys. If his profit on each chicken is \$1, on each duck is \$2 and on each turkey is \$$t$, how many of each should be raised in order to maximize his profits? Consider the cases $t = 2, 3$.

Let the number of chickens be x, the number of ducks be y, so that the number of turkeys is $200 - x - y$. Sketch the convex region in the xy-plane which corresponds to all the restrictions and then evaluate the profit at each corner point.

5. Solve the problem corresponding to problem 4, except the limitation to 100 ducks is removed but the limitation is imposed that the number of turkeys must not exceed 50. Keep x and y as in problem 4. Consider the cases $t = 2, 3$.

6. Solve the problem corresponding to problem 4, retaining the limitation on the number of ducks and imposing the additional restriction that the number of turkeys must not exceed 50. Consider the cases $t = 1, 2$.

7. (a) Verify the details of the table given in the solution of Example 8–24, and (b) give a diagram, similar to Fig. 8–18, showing that the maximum profit $P = 2x + y$ occurs at the point B.

8. A manufacturer wishes to produce two commodities in quantities x and y. Each commodity requires material, labor, and equipment use in amounts given by the following table. The number of units available is also specified.

	x	y	*Available*
Material	1	1	6
Labor	1	2	10
Equipment	2	1	10

Find the maximum profit, if the profit function is (a) $2x + 3y$, (b) $3x + 2y$.

9. Find the maximum value of $F = 3x + 4y$, subject to the restrictions

$$4x - 3y \leqq 8, \qquad 2x + 4y \leqq 8, \qquad x \geqq 0, \qquad y \geqq 0,$$

by the corner method and by the elimination method.

10. Find the maximum and minimum values of $F = 3x + 4y$, subject to the restrictions

$$-x + 3y \leqq 9, \qquad 3x + 2y \geqq 6, \qquad 5x - 2y \leqq 20, \qquad x \geqq 0, \qquad y \geqq 0,$$

by the corner method and by the elimination method.

CHAPTER 9

EXPONENTS, RADICALS, AND LOGARITHMS

9–1 Positive and negative integral exponents. In Sections 2–5 and 3–4, it was shown that if a and b are any real numbers and if n, m, k are positive integers, then

$$a^n a^m = a^{n+m}, \tag{9–1}$$

$$\frac{a^n}{a^m} = \begin{cases} a^{n-m} & \text{if} \quad n > m \\ 1 & \text{if} \quad n = m, \\ 1/a^{m-n} & \text{if} \quad n < m \end{cases} \tag{9–2}$$

$$(a^n)^k = a^{nk}, \tag{9–3}$$

$$(ab)^n = a^n b^n, \tag{9–4}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \tag{9–5}$$

It is the purpose of this and the next two sections to extend the meaning of a^n so that n may be any real number and the above laws remain valid. In this the real numbers a and b are restricted to be positive and, without loss of generality, to be greater than 1. Not all the proofs are given here, and, in particular, these laws are assumed for irrational exponents. This is essential in order to develop the theory of logarithms.

DEFINITION 9–1.

$$a^0 = 1. \tag{9–6}$$

It is readily verified that Eqs. (9–1) through (9–5) are valid if n, m, or k is replaced by zero.

DEFINITION 9–2. If p is a positive integer,

$$a^{-p} = \frac{1}{a^p}. \tag{9–7}$$

Consider the product $a^n a^m$, where $m = -p$ is a negative integer. Then

$$a^n a^m = a^n a^{-p} = \frac{a^n}{a^p},$$

where n and p are positive integers. To apply Eq. (9–2) for this case,

there are three possibilities:

$$\text{(i) } n > p, \qquad \frac{a^n}{a^p} = a^{n-p} = a^{n+m}.$$

$$\text{(ii) } n = p, \qquad \frac{a^n}{a^p} = 1 = a^0 = a^{n-p} = a^{n+m}.$$

$$\text{(iii) } n < p, \qquad \frac{a^n}{a^p} = \frac{1}{a^{p-n}} = a^{n-p} = a^{n+m}.$$

Hence Eq. (9–1) is verified if m is a negative integer.

Since Eq. (9–1) is valid for negative as well as positive integral m, Definition (9–2) makes Eq. (9–2) extraneous, but it is convenient to retain it.

Consider Eq. (9–3) when $k = -p$, and p is a positive integer. Then

$$(a^n)^k = (a^n)^{-p} = \frac{1}{(a^n)^p} = \frac{1}{a^{np}} = a^{-np} = a^{nk}.$$

Hence Eq. (9–3) is verified in this case.

In a similar manner, Eqs. (9–1) through (9–5) can be verified in the other cases where n, m, k are negative as well as positive integers.

EXAMPLE 9–1.

$$\frac{a^4}{a^6} = a^4a^{-6} = a^{-2};$$

$$a^{-3}a^{-4} = a^{-7} = \frac{1}{a^7}; \qquad (a^{-3})^2 = a^{-6} = (a^2)^{-3} = \frac{1}{a^6}.$$

9–2 Rational exponents and radicals. If a is a positive number and q is a positive integer, the positive number x such that

$$x^q = a \tag{9–8}$$

is called the *principal* qth *root* of a. It is assumed that there is one and only one such number x and this is represented by $\sqrt[q]{a}$. Thus $\sqrt[q]{a}$ is a positive number such that $(\sqrt[q]{a})^q = a$. In general, the number x is irrational. An approximation to x can be found by various methods, some of which involve the use of logarithms.

DEFINITION 9–3. If a is a positive number and q is a positive integer, then

$$a^{1/q} = \sqrt[q]{a}.$$

This definition is consistent with Eq. (9–3) for $n = 1/q$ and $k = q$, since

$$(a^{1/q})^q = (\sqrt[q]{a})^q = a \qquad \text{and} \qquad q(1/q) = 1.$$

If c is a negative number, the qth root of c is defined only if q is an *odd* integer. For if q were even, x^2 and hence $x^{2q'}$ would be positive. If c is a negative number and q is a positive odd integer, then

$$c^{1/q} = \sqrt[q]{c} = -\sqrt[q]{-c}, \quad (c \text{ negative}, q \text{ odd}). \tag{9-9}$$

For example, $\sqrt{-27}$ is not defined in terms of real numbers and

$$\sqrt[3]{-27} = -\sqrt[3]{27} = -3.$$

DEFINITION 9–4. If a is a positive number and p and q are positive integers, then $a^{p/q}$ is defined by

$$a^{p/q} = (a^{1/q})^p. \tag{9-10}$$

THEOREM 9–1. *If a is a positive number and p and q are positive integers, then*

$$(a^p)^{1/q} = (a^{1/q})^p. \tag{9-11}$$

Proof. Let $x = (a^p)^{1/q}$, so that in accord with Definition 9–3,

$$x^q = a^p = A.$$

Let $y = (a^{1/q})^p$, so that from the definition $y^{1/p} = a^{1/q}$. If both sides of this equation are raised to the power pq and Eq. (9–3) is used with n a unit fraction,

$$y^q = a^p = A.$$

Hence $A = x^q = y^q$, and because of the uniqueness assumption, $x = y$; the theorem is established.

It is now possible to prove that the fundamental laws given by Eqs. (9–1), (9–3), and (9–4) are valid for any rational numbers, but the proofs are not given here. It is assumed* that these equations are valid for all rational numbers under Definitions 9–1 through 9–4.

EXAMPLE 9–2.

$$a^{2/3}a^{1/2} = a^{2/3+1/2} = a^{7/6};$$

$$a^{2/3}a^{-1/2} = a^{2/3-1/2} = a^{1/6}; \qquad (a^{1/3})^{1/2} = a^{1/6} = \sqrt[6]{a},$$

provided $a \geqq 0$. If a were negative, $a^{1/3}$ would be a negative number, but the square root of this number would not be real.

---

* A few special cases appear as exercises.

9–3 Irrational exponents. Meaning has been given to the expression $a^{p/q}$, when a is positive and $n = p/q$ is a rational number. To give meaning to a^n when n is an irrational number, it is recalled that an irrational number n can be approximated by a sequence of rational numbers which always increase but is bounded. This sequence has a least upper bound which equals the irrational number. Suppose r and s are any two rational numbers in the sequence. The following theorem gives a useful property of r and s.

THEOREM 9–2. *If a is a number greater than 1 and if r and s are rational numbers, then $a^r < a^s$ if and only if $r < s$.*

Proof. (1) If a is positive, then $a > 1$, $a = 1$ or $0 < a < 1$. If p is a positive integer and

$$\text{if} \quad a > 1, \qquad \text{then} \qquad a^p > 1;$$

$$\text{if} \quad a = 1, \qquad a^p = 1;$$

$$\text{if} \quad 0 < a < 1, \qquad \text{then} \qquad 0 < a^p < 1.$$

These results follow from the laws of inequalities (Section 7–6, especially Theorem 7–21).

(2) If $a > 1$, then $a^{p/q} > 1$ for any positive integers p and q, that is, $a^n > 1$ if n is rational. This is proved by the method of contradiction as follows.

$$\text{If} \quad a^{p/q} = 1, \qquad \text{then} \qquad a^p = 1^q = 1,$$

which is false in view of (1) above.

$$\text{If} \quad a^{p/q} < 1, \qquad \text{then} \qquad a^p < 1^q < 1,$$

which is also false. Hence if

$$a > 1, \qquad a^{p/q} > 1.$$

(3) Let r and s be two rational numbers, so that $s - r$ is positive and rational if and only if $s > r$. Then

$$a^{s-r} > 1,$$

$$\frac{a^s}{a^r} > 1,$$

$$a^s > a^r \qquad \text{or} \qquad a^r < a^s.$$

The proof depends definitely on the hypothesis $a > 1$. If $a < 1$, the following Corollary can be proved.

COROLLARY. *If a is a positive number less than* 1, *and if r and s are rational numbers, then $a^r > a^s$ if and only if $r < s$.*

It is in the sense of this theorem that meaning is given to a^n when n is an irrational number. If r and s are decimal approximations to n such that $r < s$, then $a^r < a^s$ if $a > 1$, so that as r increases toward n, a^r increases toward a least upper bound which defines a^n. More generally, if a is any positive number different from 1, and if n is any irrational real number, then it is assumed there is a unique positive number N such that $a^n = N$, this being true in the sense that when r is a rational approximation to n, then a^r is an approximation to N.

Conversely, it is assumed that if N is any positive number, then the equation

$$a^x = N, \qquad (a > 0, a \neq 1) \tag{9–12}$$

has a unique solution x (usually irrational).

EXAMPLE 9–3. (a) $2^{\sqrt{2}}$ is a number N, such that $2^{1.414} < N < 2^{1.415}$, since $1.414 < \sqrt{2} < 1.415$.

(b) The equation $3^x = 5$ has a unique solution x such that $1.465 < x < 1.466$, since $3^{1.465} < 5 < 3^{1.466}$.

Justification for these inequalities is given in the section on logarithms and numerical calculations (Section 9–6).

Finally, it is assumed that the laws of exponents, Eqs. (9–1) through (9–5), are valid for irrational exponents.

The properties of exponents may all be summarized in the graph of the equation $y = a^x$, $(a > 1)$, which defines y as a function of x. This also defines x as a function of y because of the 1–1 correspondence between x

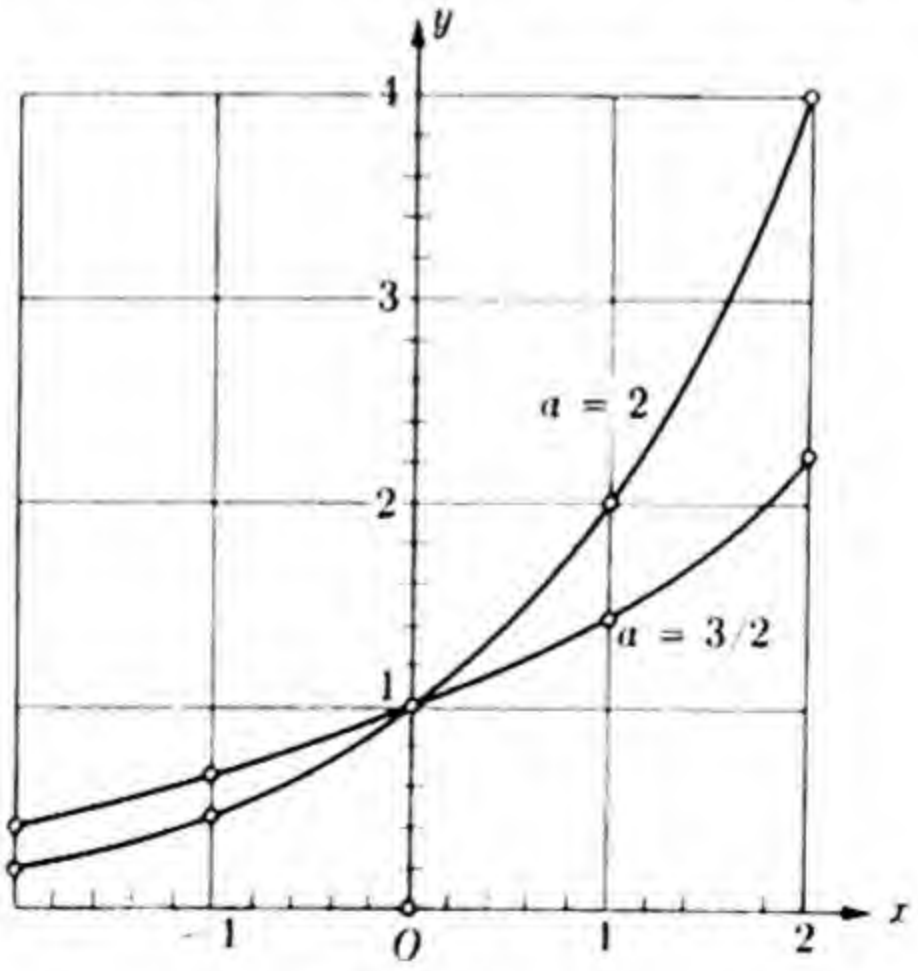


FIGURE 9–1

and y. The domain of x is the set of all real numbers and the range of y is the set of all positive real numbers. If $x = 0$, then $y = a^0 = 1$. As x increases, y increases (Theorem 9–2); and by making x sufficiently large, y can be made to exceed any given number B. If x is negative, $y = a^{-x'} = 1/a^{x'}$ is always positive; if $x' = -x$ becomes very large, y tends to 0; hence the x-axis is a horizontal asymptote. Figure 9–1 shows the graphs of $y = 2^x$ and $y = (3/2)^x$ obtained by plotting a few points and using the properties just mentioned. If $a = 10$, it would be desirable to select a smaller y-unit. The graph of the curve $y = Aa^{-x}$ could be obtained in a similar way. Since y is always monotonically decreasing, such curves are frequently used as demand curves with the form $p = Aa^{-x}$, $(0 < x < b;\ a > 1)$.

PROBLEM SET 9–1

1. Simplify the following expressions, writing the final answers with positive exponents only.

(a) $x^{-3}x^{-2}$	(b) x^{-3}/x^{-2}	(c) $x^{-3}x^{3}$	(d) x^{-2}/x^{2}
(e) $(x^{-3})^{2}$	(f) $(x^{2})^{-3}$	(g) $(x^{-2})^{-3}$	(h) $(x^{-2}x^{3})^{-3}$
(i) $(x^{2}y^{-2})^{3}$	(j) $(x^{2}/y^{-2})^{-3}$	(k) $(x^{-2}/y^{3})^{-2}$	

2. Prove (a) $a^n a^m = a^{n+m}$, if $n = 0$ and m is a positive integer; (b) $(a^n)^k = a^{nk}$, if $n = 0$ and k is a positive integer.

3. Prove (a) $a^n a^m = a^{n+m}$, if n and m are both negative integers; (b) $(a^n)^k = a^{nk}$, if n and k are both negative integers.

4. Write each of the following radicals in two other forms involving $(a^{1/q})^p$ and $(a^p)^{1/q}$. Write the expressions of parts (f), (g), (h) in radical form. Simplify the expression if possible.

(a) $\sqrt[3]{27^2}$	(b) $\sqrt[4]{16^3}$	(c) $\sqrt{(\frac{3}{4})^3}$	(d) $\sqrt[3]{a^6b^2}$
(e) $\sqrt{\frac{a^3}{b^2}}$	(f) $(a^{1/2})^3$	(g) $(a^{1/2})^3(b^{1/2})^5$	(h) $(a^2b^6)^{1/3}$

5. Simplify the following expressions; write the final answers with positive exponents only.

(a) $x^{1/2}x^{1/3}$	(b) $x^{1/3}x^{1/2}$	(c) $(x^{-1/3})^{3/2}$
(d) $(x^{-1/3})^{-3/2}$	(e) $\dfrac{2x}{3(x^{2/3})^2}$	(f) $\dfrac{-1x^{-2}}{3(x^{-1/3})^2}$

6. Prove $a^n a^m = a^{n+m}$, if $n = 1/q$, $m = 1/r$, q, and r are positive integers. (Suggestion: Let $a^{1/q} = x$, $a^{1/r} = y$ and raise both sides to the (qr)th power.)

7. Prove $(ab)^{1/q} = a^{1/q}b^{1/q}$. (Suggestion: See the proof that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ in Section 6–3.)

8. Prove the Corollary to Theorem 9-2: If $0 < a < 1$, r and s rational numbers, then $a^r > a^s$ if and only if $r < s$.

9. Sketch the curves $y = 3^x$ and $y = 3^{-x}$, $(-2 \leqq x \leqq 2)$ in the same diagram.

10. Sketch the demand curve $p = 20(3^{-x})$ and the supply curve $p = 5 + 3x$ on coordinate-ruled paper in the same diagram, and estimate where they intersect.

9–4 Logarithms. DEFINITION 9–5. If a is a positive number different from 1 and

$$\text{if} \quad N = a^n, \qquad \text{then} \qquad n = \log_a N. \tag{9-13}$$

The assumptions made in Section 9–3 are equivalent to the statements that every positive number N has a unique logarithm n, which is positive, negative, or zero; and, conversely, every real number n, is the logarithm of a positive number N. Negative numbers do not have logarithms as defined by Definition 9–5. The positive number a, $(a \neq 1)$, is called the base of the system of logarithms. The fundamental laws of logarithms are independent of the base. Several bases are in common use, the best known being the irrational number $e = 2.71828\ldots$ and the number 10. High-speed computing machines have made the base 2 quite useful. The base 10 is convenient for numerical calculations, and it is the one considered in detail in this text. Logarithms whose base is 10 are called common logarithms, and the base is omitted when no misunderstanding can occur. Thus, if

$$N = 10^n, \qquad n = \log N.$$

There are three fundamental laws of logarithms which correspond to the first three laws of exponents.

$$\log (NM) = \log N + \log M. \tag{9-14}$$

This law can be extended to any number of factors and stated as follows:

The logarithm of a product is the sum of the logarithms of the factors.

$$\log \frac{N}{M} = \log N - \log M. \tag{9-15}$$

The logarithm of a quotient is the difference of the logarithms of the numerator and denominator.

$$\log N^k = k \log N. \tag{9-16}$$

The exponent k may be a positive or negative integer, a unit fraction, a rational number or an irrational number. In all of these cases k is considered as the *power* of N; the law can be stated as follows:

The logarithm of the power of a number is the power times the logarithm of the number.

Proof. Let

$$N = 10^n, \qquad M = 10^m$$

so that

$$n = \log N, \qquad m = \log M.$$

Then

$$NM = 10^n 10^m = 10^{n+m}, \qquad \frac{N}{M} = 10^{n-m}, \qquad N^k = 10^{kn}.$$

If Definition 9-5 is applied to each of the foregoing equations, we obtain:

$$\log (NM) = n + m = \log N + \log M, \tag{9-14}$$

$$\log \frac{N}{M} = n - m = \log N - \log M, \tag{9-15}$$

$$\log N^k = kn = k \log N. \tag{9-16}$$

Algebraic expressions which involve only multiplication, division, and raising to a power can be written in logarithmic form by successive applications of those laws.

EXAMPLE 9-4. Determine the logarithmic equivalent of the equation

$$x = \sqrt[3]{A^2/B}\,.$$

When written with fractional exponents, this becomes

$$x = \frac{A^{2/3}}{B^{1/3}},$$

so that

$$\begin{aligned} \log x &= \log A^{2/3} - \log B^{1/3} \\ &= \tfrac{2}{3} \log A - \tfrac{1}{3} \log B. \end{aligned}$$

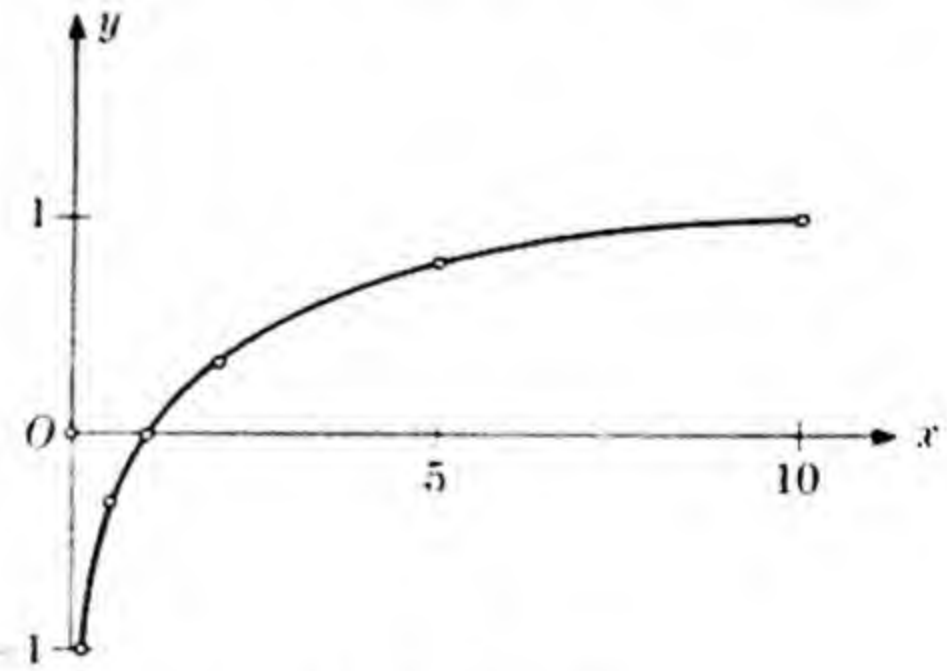


FIGURE 9-2

If $\log A$ and $\log B$ are known, $\log x$ can be computed.

The curve $y = \log x$ has a relatively simple form (Fig. 9-2). Different units have been selected on the x- and y-axes. Since $10 = 10^1$, $1 = 10^0$, $1/10 = 10^{-1}$, then $\log 10 = 1$, $\log 1 = 0$, $\log 1/10 = -1$. These three points, together with the fact that x must always be positive and that y increases as x increases [Theorem (9-2), which is also true even if r and s are irrational], enable one to make a fair sketch. The values of $\log 2$, $\log 3$, and $\log 5$, if known, can be used for more accuracy.

9–5 Tables of common logarithms. The logarithms of most integers are irrational numbers. For example: if $3 = 10^x$, then x is the logarithm of 3 and cannot be the quotient of two integers. $3 = 10^{p/q}$ would imply $3^q = (2 \cdot 5)^p$, p and q integers; and this is impossible, since the left number is divisible by 3 and the right number is not. More generally, if an integer N is such that $N = 10^{p/q}$, then $N^q = 10^p$, and N must be a power of 10. The same remark applies if N is a finite decimal fraction, for $N = M/10^m$, where M and m are integers, gives $\log N = \log M - m$, and so, in general, $\log N$ cannot be rational. Such numbers as $\sqrt{10}$, $\sqrt[3]{10}$, have rational logarithms, but these numbers are themselves not rational. When the logarithm of an integer is not rational, it can only be approximated by a rational number. This approximation consists of an integer called the *characteristic* and a pure decimal fraction called the mantissa. When the base is 10, "Tables of Logarithms" give mantissas only. This text contains a four-place table (Table II, Appendix III) for computations involving numbers of four significant figures. The characteristic is determined by rules now to be established.

Since $1 = 10^0$, $10 = 10^1$, and since the $\log M$ increases as M increases, the logarithm of a number M between 1 and 10 consists entirely of a decimal fraction with 0 as its characteristic. The decimal part is tabulated. One may read directly from the table: $\log 2 = 0.3010$; $\log 7 = 0.8451$. When any number M between 1 and 10 is expressed as a decimal fraction, it has one and only one digit to the left of the decimal point. For example, 3, 3.69, 3.697, $\sqrt{2} = 1.414\ldots$, $\pi = 3.142^-$ are numbers of this type. If M is such a number and N is given by $N = M \cdot 10^k$, k a positive integer, then N has $k + 1$ digits to the left of the decimal point and

$$\log N = \log (M \cdot 10^k) = k + \log M.$$

Hence the characteristic of $\log N$ is k, and it is noted that k is one less than the number of digits in N which precede the decimal point. The general rule is stated as follows:

If a number is greater than 1, *its characteristic is positive (or zero) and is one less than the number of digits to the left of the decimal point.*

For example: Table II shows the mantissa corresponding to the sequence of digits 369, irrespective of the decimal point, to be .5670. Hence

$$\log 3.69 = 0.5670, \qquad \log 36.9 = 1.5670, \qquad \log 36900. = 4.5670.$$

Let $N = M/10^k = M \cdot 10^{-k}$, where M is a number between 1 and 10, and k is a positive integer. The number N is a decimal fraction which has $k - 1$ zeros between the decimal point and the first nonzero digit and

$$\log N = \log M - k.$$

Hence the characteristic of N is $-k$, where k is one more than the number of zeros which follow the decimal point and precede the first nonzero digit. The general rule is stated as follows:

If a number is between 0 *and* 1, *its characteristic is negative with absolute value one greater than the number of zeros between the decimal point and the first nonzero digit.*

For example, $0.369 = 3.69/10$, so that

$$\log 0.369 = \log 3.69 - 1 = 0.5670 - 1; \qquad \log 0.00369 = 0.5670 - 3.$$

The mantissa is always positive but less than 1; thus the logarithm of any number between 0 and 1 is intrinsically negative (see Fig. 9–2). In certain problems related to the logarithmic form of an exponential law or a power law, this intrinsic value is needed. Thus $\log 0.369 = 0.5670 - 1 = -0.4330$. The decimal fraction -0.4330 is not a mantissa but is the true value of $\log 0.369$. Similarly, $\log 0.00369 = 0.5670 - 3 = -2.4330$, where the last form is the *intrinsic* value of the logarithm.

In problems involving numerical calculations, it is convenient to adjust the form of a negative characteristic by adding and subtracting the same integer so that the characteristic appears in the form of a positive integer minus 10. For example,

$$\log 0.369 = 0.5670 - 1 = 9.5670 - 10$$

$$\log 0.00369 = 0.5670 - 3 = 7.5670 - 10.$$

The digit 7 is 9 minus the number of zeros (2) which immediately follow the decimal point. The alternative rule is stated as follows:

If a number is between 0 *and* 1, *its characteristic is of the form* $c - 10$, *where* c *is* 9 *minus the number of zeros between the decimal point and the first nonzero digit.*

In numerical calculations there are two problems to solve: (1) given a number N, find $\log N$; (2) given $\log N$, find N. The mantissa depends only upon the sequence of digits and does not depend upon the position of the decimal point. For the table in this text, if N is given as a three-digit number, find the first two nonzero digits in the N column, and read the mantissa on this horizontal line and in the column headed by the third digit. The mantissa for a four-digit number is found by linear interpolation in accord with the principle of proportional parts:

$$\frac{\text{fourth digit}}{10} = \frac{\text{partial difference (p.d.)}}{\text{tabular difference (t.d.)}}. \tag{9–17}$$

The partial difference is computed to the nearest whole unit and added to the smaller mantissa. The characteristic is then determined by the rules given above.

EXAMPLE 9–5. Find the logarithms of 2437 and 0.002437.

$$\left.\begin{array}{l}\log 2.43 = 0.3856\\ \log 2.437 = 0.3869\end{array}\right]13\Bigg]18 \qquad \text{p.d.} = 18(0.7) = 12.6 = 13.$$
$$\log 2.44 = 0.3874 \;\Big]$$

The tabular difference found by subtracting the two adjacent tabular entries is 18; the partial difference is $18(0.7) = 13$, which added to log 2.43 gives log 2.437. Hence

$$\log 2437 = 3.3869; \qquad \log 0.002437 = 0.3869 - 3 = 7.3869 - 10.$$

(2) If $\log N$ is given, locate the mantissa between two successive entries in the body of the table. The first two digits of the number N are found in the column on the left of the page and in the row corresponding to the smaller entry; the third digit of N is found at the head of the column corresponding to this smaller entry, and the fourth digit is the nearest unit found by interpolation from Eq. (9-17). The decimal point is then located by reversing the rules for finding the characteristic as given above.

If the characteristic is positive (or zero), the number is greater than 1 *and the number of digits to the left of the decimal point is one more than the characteristic.*

If the characteristic is negative, the number is between 0 *and* 1, *and the number of zeros between the decimal point and the first nonzero digit is one less than the absolute value of the characteristic.*

If the characteristic is given in the form $c - 10$, *then the number of zeros between the decimal point and the first nonzero digit is* 9 *minus* c.

EXAMPLE 9–5 *continued.* If $\log N = 7.3869 - 10$, is given, the mantissa is located between the entries 3856 and 3874 corresponding to the sequences of digits 243 and 244. The fourth digit is found from the proportion

$$\frac{d}{10} = \frac{\text{partial difference}}{\text{total difference}} = \frac{13}{18} \qquad \text{or} \qquad d = 7.$$

Since the characteristic is $7 - 10$, the number of zeros immediately following the decimal point is $9 - 7 = 2$. Hence, if

$$\log N = 7.3869 - 10, \qquad N = 0.002437.$$

Similarly, if $\log N = 3.3869$, $N = 2437$, and if

$$\log N = 0.3869, \qquad N = 2.437.$$

PROBLEM SET 9–2

1. (a) What is $\log_2 N$ if $N = 8$; if $N = 1/16$?
   (b) What is N if $\log_2 N = 4$; if $\log_2 N = -3$?
   (c) What is $\log N$ if $N = 1000$; if $N = 0.001$?
   (d) What is N if $\log N = 4$; if $\log N = -4$?

2. Write the logarithmic equations which correspond to

(a) $x = \dfrac{AB}{C}$; (b) $x = \sqrt{\dfrac{A}{C}}$; (c) $x = \sqrt[3]{\dfrac{AB^2}{C}}$.

3. If $y = \log_2 x$, write the equation which gives x in terms of y. Draw the corresponding curve for $-2 < y < 2$ by assigning values to y and computing x.

4. Find the common logarithms of the following numbers.
   (a) 423; 7610; 0.0287
   (b) 87900; 87.9; 0.00879
   (c) 4237; 7689; 0.02873
   (d) 87960; 87.92; 0.008798
   (e) 6.893; 0.7642; 1.023

5. Find the numbers which have the following common logarithms.
   (a) 2.2765; 0.2765 − 2; 7.2765 − 10
   (b) 3.8960; 0.8960 − 1; 8.8960 − 10
   (c) 0.2935; 0.2935 − 3; 9.2935 − 10
   (d) 1.4276; 0.4276 − 1; 7.4276 − 10
   (e) 2.7180; 0.7180 − 2; 9.7180 − 10

9–6 Numerical calculations. If a product, quotient, or the power of a number is to be computed, or if the solution of an exponential or power equation is to be computed, the first step is to write the logarithmic equivalent of the given equation. Before using the tables, a complete logarithmic skeleton is planned from this equation. This skeleton is the part of the computation which ordinarily appears to the left of the equal signs.

EXAMPLE 9–6 (compare Example 9–4). Compute $x = \sqrt[3]{A^2/B}$, where A and B are given numbers.

The logarithmic equation is $\log x = (2 \log A - \log B)/3$.

The logarithmic skeleton is:

$$\begin{aligned} \log A &= \\ 2 \log A &= \\ (-) \quad \log B &= \\ \hline 3\overline{)\qquad\qquad} & \\ \log x &= \\ x &= \end{aligned}$$

The (−) to the extreme left indicates subtraction.

EXAMPLE 9-7. Compute

$$x = \sqrt{\frac{8.968 \times 347.0}{4321}}.$$

$$\log x = (\log 8.968 + \log 347.0 - \log 4321)/2$$

$$\log 8.968 = 0.9527$$

$$(+)\quad \log 347.0 = 2.5403$$

$$23.4930 - 20$$

The characteristic is adjusted for the subtraction and division by 2

$$(-)\quad \log 4321 = 3.6356$$

$$2\overline{)19.8574 - 20}$$

$$\log x = 9.9287 - 10$$

$$x = 0.8486.$$

EXAMPLE 9-8. Compute $x = \sqrt[3]{-10}$.

Negative numbers have no logarithms, so this equation is written

$$x = -\sqrt[3]{10} \qquad \text{or} \qquad (-x) = \sqrt[3]{10}.$$

Then

$$\log (-x) = \tfrac{1}{3} \log 10 = 0.3333$$

$$(-x) = 2.154 \qquad 10\,\frac{\text{p.d.}}{\text{t.d.}} = \frac{90}{21} = 4.$$

$$x = -2.154$$

EXAMPLE 9-9. Compute $x = \sqrt{36 + \sqrt{250}}$.

First compute $\sqrt{250}$ and then add this to 36. The scheme is as follows:

$$\log 250 = 2.3979$$

$$\tfrac{1}{2} \log 250 = 1.1990 \qquad \text{Computer's rule}$$

$$\sqrt{250} = 15.81 \qquad 10(\tfrac{3}{27}) = 1$$

$$36$$

$$S = 51.81$$

$$\log S = 1.7144$$

$$\tfrac{1}{2} \log S = 0.8572$$

$$x = 7.198$$

EXAMPLE 9-10. Compute $2^{\sqrt{2}}$ approximately (see Example 9-3) using a four-place table of logarithms.

$$2^{1.414} < 2^{\sqrt{2}} < 2^{1.4142}.$$

Since only a four-place table is available, $2^{\sqrt{2}}$ is approximated by $2^{1.414}$:

$$\log 2 = 0.3010$$
$$\sqrt{2}\log 2 = (0.3010)(1.414) = 0.4256$$
$$2^{\sqrt{2}} = 2.664 \qquad d = 10(\tfrac{7}{16}).$$

The equation

$$y = A \cdot 10^{kx} \tag{9-18}$$

in logarithmic form becomes

$$\log y = \log A + kx.$$

If a new variable Y, $Y = \log y$, is introduced, and if $\log A = a$, the equation takes the linear form

$$Y = a + kx \tag{9-18L}$$

with graph in the xY-plane as a straight line. If x is given a specific value, the value of y is computed by finding Y and getting y from this.

EXAMPLE 9–11. Study the graph of $y = 3 \cdot 10^{x/2}$, $(0 < x < 4)$, and the graph of its logarithmic form. If $x = 3$, compute y.

In the graph of the curve $y = 3 \cdot 10^{x/2}$, $(0 < x < 4)$, the range of y is between 3 and 300 and the y unit must be very small if much of the curve is to be shown. The logarithmic form is

$$\log y = \log 3 + \tfrac{1}{2}x.$$

If $x = 3$,

$$\log y = 0.4771 + 1.5000 = 1.9771$$
$$y = 94.85.$$

If $Y = \log y$ and $\log 3$ is rounded off to 0.48, this equation becomes $Y = 0.48 + x/2$; the range of Y is roughly $\frac{1}{2}$ to $\frac{5}{2}$, the graph is a straight line (Fig. 9–3) and can be constructed from the two points $(0, \frac{1}{2})$, $(4, \frac{5}{2})$. The third point (3, 2) can be used as a check.

The equation

$$y = Ax^k \tag{9-19}$$

in logarithmic form is

$$\log y = \log A + k \log x.$$

If new variables $X = \log x$, $Y = \log y$ are introduced, the equation takes

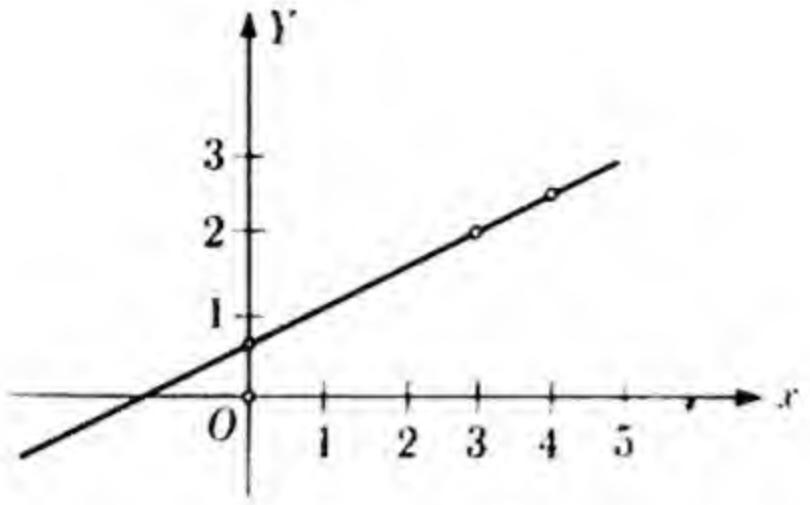


FIGURE 9-3

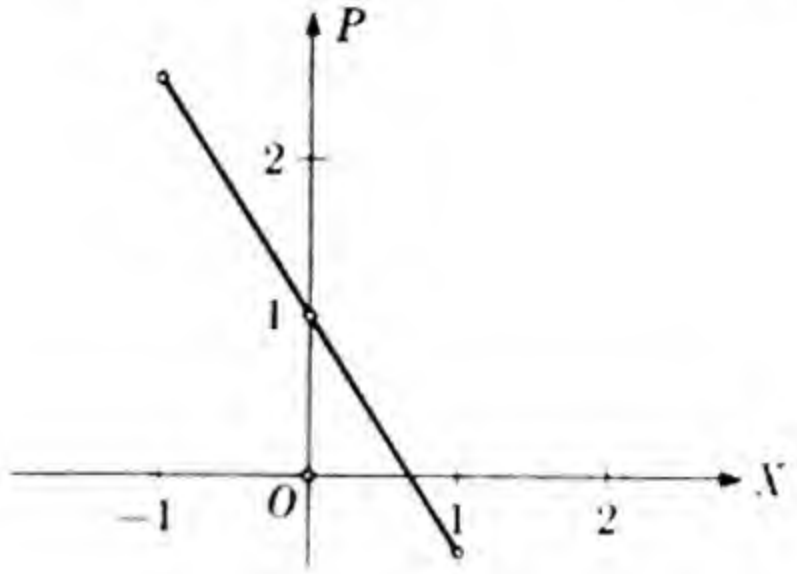


FIGURE 9-4

the linear form

$$Y = a + kX \tag{9-19L}$$

with graph in the XY-plane which is a straight line. If x is given a specific value, the value of y is computed by means of logarithms.

EXAMPLE 9-12. A demand law is given in the form $p = 10/x^{3/2}$, $(\frac{1}{10} < x < 10)$. Write the logarithmic form of this equation, introduce new variables $P = \log p$, $X = \log x$, and draw the corresponding line in the PX-plane. If $x = 5$, compute p.

$$\log p = \log 10 - \tfrac{3}{2} \log x$$
$$P = 1 - \tfrac{3}{2}X, \qquad (-1 < X < 1),$$

since $\log 10 = 1$ and $\log \frac{1}{10} = -1$. The corresponding line segment is obtained from the points $(-1, \frac{5}{2})$, $(1, -\frac{1}{2})$ and is shown in Fig. 9-4. The P-intercept may be used as a check.

If $x = 5$, $\log p = 1 - \frac{3}{2}(0.6990) = -0.0485$. This is the intrinsic value of the logarithm. When expressed in terms of a positive mantissa,

$$\log p = 9.9515 - 10, \qquad p = 0.8944.$$

The exponential equation

$$a^x = b, \tag{9-20}$$

where a and b are given positive constants, has the logarithmic form

$$x \log a = \log b$$

or

$$x = \frac{\log b}{\log a}. \tag{9-20L}$$

If a or b is between 0 and 1, its logarithm is negative and its intrinsic value is used. The actual division of $\log a$ by $\log b$, except for a possible negative sign, could also be made using logarithms.

EXAMPLE 9-13. Solve each of the following exponential equations.

$$(1)\ 2^x = 6; \qquad (2)\ (0.2)^x = 6; \qquad (3)\ (0.2)^x = 0.6.$$

$$\log 2 = 0.3010; \qquad \log 0.2 = 9.3010 - 10 = -0.6990$$

$$\log 6 = 0.7782; \qquad \log 0.6 = 9.7782 - 10 = -0.2218$$

$$x_1 = \frac{7782}{3010}; \qquad x_2 = -\frac{7782}{6990}; \qquad x_3 = \frac{2218}{6990}$$

$$x_1 = 2.585; \qquad x_2 = -1.163; \qquad x_3 = 0.3173$$

The computation for x_2 is as follows:

$$-x_2 = \frac{7782}{6690}$$

$$\log(-x_2) = \log 7782 - \log 6990.$$

$$\log 7782 = 3.8911$$

$$(-) \qquad \log 6690 = \underline{3.8254}$$

$$\log(-x_2) = 0.0657$$

$$-x_2 = 1.163 \qquad \frac{120}{37} = 3$$

$$x_2 = -1.163.$$

PROBLEM SET 9-3

(Make all computations using a four-place table of logarithms.)

1. (a) $423 \times 761 \times 287$ (b) $(423 \times 761)/287$
   (c) $8.79 \times 2.87 \times 0.0423$ (d) $(87.9 \times 28.7)/42300$
   (e) $\dfrac{879 \times 287}{761 \times 423}$ (f) $\dfrac{0.0879 \times 2.87}{(4.23)^2}$

2. (a) $42.37 \times 0.07614 \times 2.873$ (b) $\dfrac{42.37 \times 0.07614}{2.873}$
   (c) $8.798 \times 1.894 \times 2.873$ (d) $\dfrac{8.798 \times 1.894}{2.873}$
   (e) $\dfrac{8798 \times 2873}{7614 \times 4237}$ (f) $\dfrac{0.08798 \times (2.873)^3}{(4.237)^2}$

3. (a) $\sqrt{26.78}$ (b) $\sqrt[3]{26.78}$ (c) $\sqrt{.008102}$
   (d) $\sqrt[3]{-0.15}$ (e) $\sqrt{12\sqrt[3]{100}}$ (f) $\sqrt[3]{4 - \sqrt{40}}$

4. (a) $(5.23)^{2/3}$ (b) $(5.23)^{-1/2}$ (c) $(5.23)^{\sqrt{3}}$
(d) $(-2.51)^{2/3}$ (e) $(\sqrt{3})^{\sqrt{3}}$ (f) $(1/10)^{1/10}$

5. If $y = 4 \cdot 10^{x/3}$, $(0 < x < 4)$,

(a) write the equation in logarithmic form, let $Y = \log y$, and draw the corresponding line segment in the xY-plane;

(b) if $x = 1$ and $x = 4$, compute y;

(c) if $y = 10$ and $y = 20$, compute x.

6. If $y = 30 \cdot 10^{-x/2}$, $(0 < x < 4)$,

(a) write the equation in logarithmic form. Let $Y = \log y$ and draw the corresponding line segment in the xY-plane;

(b) if $x = 1$ and $x = 3$, compute y;

(c) if $y = 10$ and $y = \frac{1}{2}$, compute x.

7. A demand law is given in the form $p = 20/x^{3/2}$, $(\frac{1}{4} < x < 4)$.

(a) Write the equation in logarithmic form. Let $X = \log x$, $P = \log p$, determine the domain of X, and draw the corresponding line segment in the XP-plane;

(b) if $x = \frac{1}{2}$ and $x = 3$, compute p,

(c) if $p = 10$ and $p = 40$, compute x.

8. A population curve is given in the form $y = 20{,}000(1 + x)^{0.32}$, where x is the number of years measured from the present and y is the population at time x.

(a) Write the equation in logarithmic form. Let $X = \log (1 + x)$ and $Y = \log y$ and draw the corresponding curve in the XY-plane.

(b) Show that the population will double in about eight years.

(c) How large will the population be in five years?

(d) When will the population be 30,000?

9. Verify that

(a) $3^{1.465} < 5 < 3^{1.466}$ (b) $2^{1.651} < \pi < 2^{1.652}$.

10. Solve the following exponential equations.

(a) $5^x = 8$ (b) $(0.5)^x = 8$ (c) $(0.5)^x = 0.8$
(d) $4^{x^2} = 1$ (e) $7.2^x = 13$ (f) $13.3^x = 7$

9–7 Compound interest. Money earned (or paid) for the use of money, periodically added to the principal and thereafter earning the same rate of interest, is referred to as *compound interest*. To specify a compound interest rate, both the per cent and the time intervals must be given. This may be done in several ways. Thus 6% compounded annually indicates that the interest is added to the principal and becomes new principal at the end of each year; $4\frac{1}{2}\%$ compounded quarterly means that at the

end of each three months $1\frac{1}{8}\%$ of the principal is added to the principal; 6% payable monthly means that $\frac{1}{2}\%$ interest is paid at the end of each month. If no period is specified, it is usually understood to be one year.

If the original principal or *present value* is P, and if the interest rate per period is i, then at the end of the first period, the new principal or *accumulated amount* A is $P + Pi = P(1 + i)$. At the end of two periods it is $A = P(1 + i)(1 + i) = P(1 + i)^2$; at the end of three periods $A = P(1 + i)^3$, and so on. At the end of n periods (n an integer),

$$A = P(1 + i)^n. \tag{9-21}$$

The derivation of this formula is an illustration of *incomplete induction*. A complete proof, based on the *Principle of Mathematical Induction*, is discussed in the next chapter. If n is not an integer, Eq. (9–21) is taken as the *definition* of the accumulated amount A of the present value P for n periods at the rate i per period.

If this equation is solved for P,

$$P = A(1 + i)^{-n} = Av^n, \tag{9-22}$$

where the symbol v stands for $(1 + i)^{-1} = 1/(1 + i)$.

If any three of the four numbers i, n, P, A are given, the others may be computed. Four-place log tables are limited to the use of four significant figures, and if n is large, the error in $\log (1 + i)^n = n \log (1 + i)$ would also be large. Hence such a table is not satisfactory for computing either A or P to the nearest cent except when n is small and the principal is less than \$100. For that reason, special tables of $(1 + i)^n$ and of $(1 + i)^{-n}$ have been prepared (Tables III and IV, Appendix III). The tabulated values are given to five decimal places, or to six significant figures, which is sufficient for principals of several thousands of dollars. If some needed n is not included in the table, the fundamental laws of exponents may be used to obtain a value.

EXAMPLE 9–14. When a boy is born, \$500 is placed to his credit in an account that pays (a) 6% annually, (b) 6% compounded quarterly, (c) 6% compounded monthly. If the account is not disturbed, what amount will there be to his credit on his twentieth birthday?

(a) $A = 500(1.06)^{20} = 500(3.20714) = \$1603.57$

(b) $A = 500(1.015)^{80} = 500(3.29066) = \$1645.33$

(c) $A = 500(1.005)^{240} = 500(1.005)^{200}(1.005)^{40}$

$= 500(2.71152)(1.22079)$

$= 500(3.31020) = \$1655.10.$

In part (c), the value of $(1 + i)^n$ could not be read directly from Table III and an auxiliary calculation was necessary.

EXAMPLE 9–15. Mr. Needy borrowed money from the E-Z Loan Co. and promised to pay \$200 at the end of one year. The interest rate is $2\frac{1}{2}\%$ per month on the first \$100 and 2% per month on the second \$100. How much cash does he receive?

$$X = \$100(1.025)^{-12} + \$100(1.02)^{-12}$$
$$= \$74.356 + \$78.849 = \$153.20 \quad \text{(Table IV)}$$

In this problem it is reasonable to use logarithms for the calculations.

$$X = X_1 + X_2, \qquad \text{where} \qquad X_1 = \frac{100}{(1.02)^{12}}, \qquad X_2 = \frac{100}{(1.025)^{12}}$$

	$\log (1.02) = 0.0086$		$\log 1.025 = 0.0107$
	$\log \ 100 = 2.0000$		$\log \ 100 = 2.0000$
$(-)$	$12 \log (1.02) = 0.1032$	$(-)$	$12 \log 1.025 = 0.1284$
	$\log X_1 = 1.8968$		$\log X_2 = 1.8716$
	$X_1 = 78.85$		$X_2 = 74.40$

$$X = \$78.85 + \$74.40 = \$153.25$$

The slight difference is due to multiplying the error in $\log (1 + i)$ by 12.

If n is the unknown, it can be calculated by interpolation in the table, although only integral values of n have realistic interpretations. It can also be calculated from the logarithmic form of Eq. (9–21):

$$n = \frac{\log A - \log P}{\log (1 + i)}. \tag{9–21L}$$

EXAMPLE 9–16. What is the least number of years required for money to double itself and what is the accumulated amount of \$100 (to the nearest dime) for this time, if the interest rate is

(a) 3% per year, (b) $3\frac{3}{4}\%$ per year?

(a) An examination of Table III under the column headed 3% shows that the entry 2.00 would occur when n is between 23 and 24. Hence the least number of years is 24, and \$100 will accumulate to \$203.3.

(b) The interest rate of $3\frac{3}{4}\%$ does not appear in the tables, but a process of *double interpolation* can be used to find an approximation to n.

The entries in the four corners of the following tabulation are found in Table III (rounded off):

n / %	$3\frac{1}{2}$	$3\frac{3}{4}$	4
18	1.857	1.942	2.026
n		2.000	
19	1.923	2.015	2.107

Since $3\frac{3}{4}$ is midway between $3\frac{1}{2}$ and 4, the values corresponding to $n = 18$ and $n = 19$ are found by interpolation to be 1.942 and 2.015. The value of n lies between 18 and 19, and at the end of 19 years the accumulated value of \$100 would be \$201.5. If a fractional value of n is desired, it can be found by interpolation to be

$$n = 18 + \frac{\text{p.d.}}{\text{t.d.}} = 18 + \frac{58}{73} = 18.7.$$

The logarithmic solution of the problem is obtained from

$$n = \frac{\log 2}{\log (1.0375)} = \frac{0.3010}{0.0160} = 18.8.$$

The log (1.0375) was found by interpolation, using $\frac{3}{4}$ of the tabular difference of log (1.03) and log (1.04). The accumulated amount of \$100 at the end of 19 years is $A = \$100(1.0375)^{19}$. The calculations appear below

$$\begin{aligned} \log 1.0375 &= 0.0160 \\ 19 \log (1.0375) &= 0.3040 \\ \log A &= 2.3040 \\ A &= \$201.4. \end{aligned}$$

In a similar way, if i is the unknown it can be calculated by interpolation, or from the logarithmic form of Eq. (9–21) written

$$\log (1 + i) = \frac{\log A - \log P}{n}.$$

In both methods the result is reliable to the nearest $\frac{1}{10}$ of 1%.

EXAMPLE 9–17. At what rate of compound interest will money triple itself in 18 years?

$$3 = (1 + i)^{18}, \qquad \log (1 + i) = \frac{\log 3}{18}.$$

(a) Table III shows that if $i = 6\%$, $(1 + i)^{18} = 2.85$; and if $i = 6\frac{1}{2}\%$, $(1 + i)^{18} = 3.11$. Linear interpolation yields

$$i = 6\% + \tfrac{15}{26} \times \tfrac{1}{2}\% = 6.3\%.$$

(b) $\log 3 = 0.4771$, $\log (1 + i) = 0.0265$, $\log (1.06) = 0.0253$; the partial difference is 12, the tabular difference is 41, so

$$(1 + i) = 1.063 \qquad \text{or} \qquad i = 6.3\%.$$

In many transactions, when the number of periods is not an integer, compound interest is used for the integral number of periods and simple interest is used for the fractional part of the period. This works very well for finding the accumulated value but is not so convenient for finding present value. In order to compare two sums of money due at different dates, it is necessary to find their values at the same date. One of the advantages of compound interest is that $(1 + i)^n(1 + i)^m = (1 + i)^{n+m}$ for all real values of n and m and the comparison date may be selected for convenience. When using simple interest, the comparison date must be specified.

EXAMPLE 9–18. (a) Find the accumulated amount of \$1000 for $6\frac{1}{4}$ years if the interest rate is 6% per year and compound interest is used for the first 6 years and simple interest for the fractional part of the years.

(b) Find the present value of \$1000 due in $6\frac{1}{4}$ years if the interest rate is 6% per year and compound interest is used for the first 6 years and simple interest for $\frac{1}{4}$ year.

(c) Compare this with the result obtained if the value is found 7 years before it is due and this is accumulated at simple interest for $\frac{3}{4}$ of a year.

$$\begin{aligned}
\text{(a)}\quad A &= \$1000(1.06)^6(1.015) \\
&= \$1000(1.41852)(1.015) = \$1439.80 \\
\text{(b)}\quad P &= (1000)(1.06)^{-6}/(1.015) \\
&= \$(1000)(0.70496)/1.015 = \$694.54 \\
\text{(c)}\quad P &= \$1000(1.06)^{-7}(1.045) \\
&= \$1000(0.66506)(1.045) = \$694.99.
\end{aligned}$$

If compound interest had been used in (b) and (c), it is easy to observe that

$$(1.06)^{-6}(1.06)^{-1/4} = (1.06)^{-7}(1.06)^{3/4} = (1.06)^{-25/4}.$$

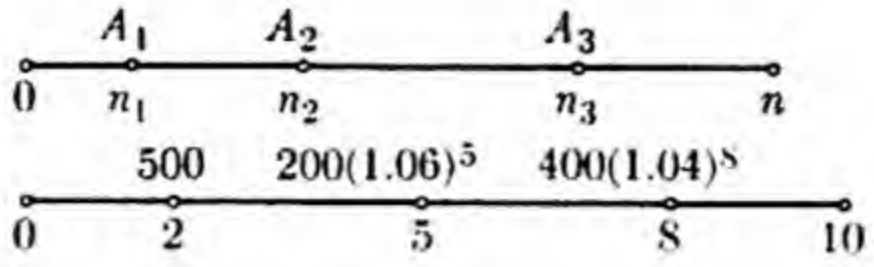


FIGURE 9–5

The difficulty lies in obtaining $(1.06)^{1/4}$ accurately. Four-place tables of logarithms do not suffice. Methods of doing this are explained in the next chapter.*

Equation of value. If several sums of money, represented by interest or noninterest bearing notes, are due at various dates, the total value at any specified date can be found by accumulating or discounting† each note to that date at a given interest rate. If the comparison date is the present, the corresponding equation is called the *equation of present value.* If the comparison date is some later date, it is called an *equation of future value,* and the comparison date must be stated. If the sums are $A_1, A_2, A_3, \ldots,$ due at times $n_1, n_2, n_3, \ldots,$ they may be the accumulated amount of an interest bearing note.

EXAMPLE 9–19. Find (a) the equation of present value, (b) the equation of value at the end of 5 years, and (c) the equation of value at the end of 10 years for the following notes: \$500 due in 2 years, \$200 due in 5 years with interest at 6% per year, and \$400 due in 8 years with interest at 4% per year. Consider that money is worth 5% per year. (See Fig. 9–5.)

(a) To find the equation of present value, \$500 must be discounted at 5% compound interest for 2 years; $\$200(1.06)^5$ must be discounted for 5 years, and $\$400(1.04)^8$ must be discounted for 8 years.

$$P = 500(1.05)^{-2} + 200(1.06)^5(1.05)^{-5} + 400(1.04)^8(1.05)^{-8}.$$

(b) To find the equation of value at the end of five years, \$500 must be accumulated for 3 years and $\$400(1.04)^8$ must be discounted for 3 years.

$$V_5 = 500(1.05)^3 + 200(1.06)^5 + 400(1.04)^8(1.05)^{-3}.$$

It is easy to verify that

$$V_5 = P(1.05)^5 = 500(1.05)^{-2+5} + 200(1.06)^5(1.05)^{-5+5} + 400(1.04)^8(1.05)^{-8+5}.$$

---

* $\sqrt[4]{1.06}$ could be found by taking the square root of 1.06 and then the square root of this result, using the division process.

† The term "discount" as used here means to find the value at a date before it is due. It should not be confused with the noun "discount" as used when discussing simple discount.

Similarly,

$$V_{10} = 500(1.05)^8 + 200(1.06)^5(1.05)^5 + 400(1.04)^8(1.05)^2.$$

and

$$V_{10} = P(1.05)^{10} = V_5(1.05)^5.$$

It is left as an exercise to use the Tables and compute P, V_5, and V_{10} and to verify numerically that $V_{10} = P(1.05)^{10}$.

Problem Set 9–4

1. Find the accumulated amount for \$600 at the end of 24 years at 4% if interest is compounded (a) annually, (b) semiannually, (c) quarterly.

2. What sum of money should be set aside in a savings account at the time of the birth of a son to provide \$1000 when he is 20 years old if the fund earns 5% compounded (a) annually, (b) quarterly, (c) monthly?

3. A student borrows \$500 and promises to repay the loan together with interest at 4% compounded annually at the end of 6 years. Two years later this note is sold to yield the purchaser 3% annually. How much was paid for the note?

4. A student borrows \$800 and promises to repay the loan, together with interest at 3% compounded annually, at the end of 6 years. This note is immediately sold to yield the purchaser 5% annually. How much was paid for the note?

5. Mr. Poor borrows money from the Short-Time Loan Company and promises to pay \$300 at the end of one year. The interest rate is 2% per month for the first \$100 and $1\frac{3}{4}$% per month for the balance. How much cash does he receive? Make the computations using Table IV and also using logarithms.

6. Mr. Needy borrowed money from the E-Z Loan Co. and promised to repay \$100 at the end of 6 months and \$100 at the end of 12 months. The interest rate on the shorter loan is $2\frac{1}{2}$% per month, and is 2% per month on the other loan. How much cash does he receive? Make the computations using Table IV and also using logarithms.

7. How long will it take money to double itself and what is the accumulated amount of \$100 (to the nearest dime) for the least number of whole periods if the interest rate is 5%, (a) compounded semiannually, (b) compounded quarterly?

8. How long will it take money to triple itself and what is the accumulated amount of \$100 (to the nearest dollar) for the least number of whole periods if the interest rate is $3\frac{3}{4}$% per year?

Solve the problem (a) by means of double interpolation, and (b) by means of logarithms.

9. At what rate of interest will \$75 accumulate to \$100 in (a) 10 years, (b) 8 years?

10. It is claimed that the government's $3\frac{3}{4}$% saving bonds increase their value by one-third in 7 years and 9 months. (a) Verify this by computing $(1.0375)^{7.75}$. (b) How long will it take for such a bond to double its value? Compute this by

determining m so that $(4/3)^m = 2$. Compare your result with that in Example 9–16.

11. At what interest rate will money triple itself in 30 years? Solve (a) by interpolation, and (b) by logarithms.

12. Find the accumulated amount of \$5000 for $10\frac{1}{2}$ years at 5% (a) if compound interest is used for the 10 years and simple interest is used for the half-year; (b) if compound interest is used throughout. Evaluate $\sqrt{1.05}$ to five places of decimals by the division process.

13. A note for \$1000 (without interest) is due in 6 months. What is its accumulated value at the end of the year if money is worth 6% per year?

(a) Show algebraically that the following procedures are equivalent: (1) accumulate \$1000 for the half-year at compound interest; (2) first find the present value of \$1000 at compound interest, then accumulate this for one year at compound interest.

(b) Show by means of numerical calculations that the following procedures give slightly different results: (1) accumulate \$1000 for the half-year at simple interest; (2) first find the present value of \$1000 at simple interest and then accumulate this for one year at simple interest.

14. (a) Write the equation of present value for the following non-interest bearing notes if money is worth $4\frac{1}{2}$% per year: \$200 due in 4 years, \$400 due in 6 years, \$250 due in 7 years. Compute this present value.

(b) Write the equation of value at the end of 10 years for the same notes and indicate the relation between the values in (a) and (b). Compute the value at the end of 10 years.

15. (a) Find the present value of the following notes if money is worth 5% per year: \$2000 (without interest) due in 1 year and \$1000 with interest at 4% per year, due in 3 years. (b) What is the value of these two notes at the end of 5 years?

16. If money is worth 5% per year, what single amount payable at the end of 6 years is equivalent to two notes: \$1000 due in 4 years at 6% per year, and \$2000 due in 10 years at 4% per year.

17. Make the indicated numerical calculations for Example 9–19, computing P, V_5, and V_{10}. Then verify that $V_{10} = P(1.05)^{10}$.

12398

CHAPTER 10

MATHEMATICAL INDUCTION, PROGRESSIONS, BINOMIAL THEOREM, AND ANNUITIES

10–1 Mathematical induction. Although many important developments have resulted from inductive reasoning, the history of mathematics is filled with examples of conclusions based on consideration of a relatively few special cases which have been found false upon further examination. Some process is needed to show that the statement arrived at by inductive reasoning from a few special cases is either always true or is false. Mathematical induction is one such process for sequences of numbers. Before stating the *Principle of Mathematical Induction*, a few preliminary remarks are made, and several illustrations are given.

A sequence of numbers $u_1, u_2, u_3, \ldots, u_n$, where u_n is some function of the positive integer n: $u_n = f(n)$, is an ordered set, that is, for each positive n there is one and only one value of u. For example, the nth odd integer can be expressed in the form $u_n = 2n - 1$. It may be verified directly that if $n = 1, 2, 3, 4, \ldots, u_n = 1, 3, 5, 7, \ldots$ all fit this formula. Suppose this formula is true for all values of n up to and including the integer k: $u_k = 2k - 1$. The next odd integer is

$$u_{k+1} = u_k + 2 = 2k - 1 + 2 = 2(k + 1) - 1.$$

This is the formula with k replaced by $k + 1$. Since the formula has been verified for $k = 1, 2, 3, 4$, it must be true for $k = 5$, then $k = 6$; and the Principle of Induction asserts that it is true for any positive integral value of n.

Another example of a sequence of numbers is formed by adding the u's: $S_1 = u_1, S_2 = u_1 + u_2, \ldots, S_n = u_1 + u_2 + \cdots + u_n = F(n)$. (See Example 10–1.)

The formula for the compound amount,

$$A = P(1 + i)^n,$$

at the rate i per period for n periods (n a positive integer), was established by incomplete induction. The result seemed to hold for all n. This formula is made precise by use of mathematical induction. Assume the formula true for all positive integer values up to the integer k:

$$A_k = P(1 + i)^k.$$

Then A_{k+1} is the result of accumulating A_k for one period, so

$$A_{k+1} = P(1 + i)^k(1 + i) = P(1 + i)^{k+1}.$$

This is of the same form as A_k except that k has been replaced by $k + 1$. The Principle of Mathematical Induction then states the formula is true for any positive integral value of n.

As another illustration consider the statement: If n is a positive integer, then $x^n - y^n$ is divisible by $x - y$. Restated,

$$x^n - y^n = (x - y)Q(x, y),$$

where $Q(x, y)$ is a polynomial in x and y.

(a) It is *verified* directly that the statement is true for $n = 1$ and $n = 2$:

$$x - y = (x - y)(1); \qquad x^2 - y^2 = (x - y)(x + y).$$

(b) *Assume* that the statement is true for all positive integers n up to and including k:

$$x^k - y^k = (x - y)Q(x, y).$$

(c) By means of *algebraic manipulations*, it is shown that $x^{k+1} - y^{k+1}$ is of the same form:

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k) \\ &= x^k(x - y) + y(x - y)Q(x, y) \\ &= (x - y)[x^k + yQ(x, y)] = (x - y)Q_1(x, y). \end{aligned}$$

This shows that the statement is then true for $n = k + 1$.

(d) The Principle of Induction implies that the statement is true for any positive integral value of n.

A proof by mathematical induction involves four essential ideas, none of which can be omitted: (1) verification; (2) assumption; (3) algebraic extension; (4) use of the Principle of Induction. The axiom of mathematical induction can be stated as follows:

> *If some rule which depends upon the value of the positive integer n can be verified for at least one fixed value of n, and if it can be shown that the assumption that the rule is true for $n = k$ implies that it is also true for $n = k + 1$, then the rule is true for any n greater than the fixed n.*

EXAMPLE 10–1. Prove that the sum of the first n odd integers is a perfect square.

It has already been proved by induction that the nth odd integer is $2n - 1$. A few trials show $S_1 = 1 = 1^2$, $S_2 = 1 + 3 = 4 = 2^2$, $S_3 = 1 + 3 + 5 = 9 = 3^2$; and it is suspected that

$$S_n = 1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

(a) It has been verified that this is true for $n = 1, 2, 3$.
(b) Assume it is true for all positive integers up to k:

$$S_k = 1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

(c) The sum of the first $(k + 1)$ odd integers is obtained by adding the odd integer $2k + 1$ to S_k, so that

$$\begin{aligned} S_{k+1} &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + 2k + 1 = (k + 1)^2, \end{aligned}$$

which is the given formula with k replaced by $k + 1$.

(d) Therefore, by the Principle of Mathematical Induction, the rule is true for any positive integer n.

The above argument can be condensed, but the essentials of the argument are required.

10–2 Arithmetic progression. DEFINITION. An *arithmetic progression* is a sequence of numbers such that each number after the first is obtained from the previous one by adding a fixed number d.

If the first term is a, the sequence is

$$a, a + d, a + 2d, a + 3d, \ldots, u_n = a + (n - 1)d.$$

The form of u_n is checked for $n = 1, 2, 3$, and since

$$[a + (n - 1)d] + d = a + nd = a + ([n + 1] - 1)d,$$

it follows by mathematical induction that

$$u_n = a + (n - 1)d \tag{10-1}$$

for any positive integer n.

The sum of the first n terms of the arithmetic progression ($n = 1, 2, \ldots$) form a second sequence

$$S_1 = u_1, \qquad S_2 = u_1 + u_2,$$
$$S_3 = u_1 + u_2 + u_3, \ \ldots, \qquad S_n = u_1 + u_2 + \cdots + u_n.$$

A formula for S_n is obtained by writing the sequence in detail, then re-

writing it in reverse order, and adding:

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + u_n$$
$$S_n = u_n + (u_n - d) + (u_n - 2d) + (u_n - 3d) + \cdots + a$$
$$2S_n = (a+u_n) + (a+u_n) + \cdots + (a+u_n) = (a+u_n)n.$$

Therefore

$$S_n = \frac{1}{2}[a + a + (n-1)\,d]n = na + \frac{(n-1)n}{2}d. \qquad (10\text{–}2)$$

If u_n is considered the last term and represented by the symbol l, an equivalent formula is

$$S_n = \frac{(a+l)n}{2}. \qquad (10\text{–}3)$$

If Eq. (10–2) is first verified for $n = 1, 2, 3, \ldots$, its validity can be established by mathematical induction.

EXAMPLE 10–2. (a) Show that Eq. (10–3) can be used to prove that the sum of the first n odd integer is n^2.

(b) Find the sum of all the integers between 1 and 100 that are not divisible by 3.

(a) $a = 1, \quad l = u_n = 2n - 1; \quad S_n = \dfrac{1 + 2n - 1}{2} n = n^2.$

(b) The numbers can be written in two sets:

$$1, 4, 7, \ldots, 97$$
$$2, 5, 8, \ldots, 98$$

which when added gives the set

$$3, 9, 15, \ldots, 195$$

whose sum is

$$3(1 + 3 + 5 + \cdots + 65).$$

Since $65 = 2n - 1$ gives $n = 33$, this sum is $3 \times 33^2 = 3267$.

If $u_1, u_2, \ldots, u_n$ is a given sequence, its *arithmetic average* $\bar{u}$ is defined as S_n/n or

$$\bar{u} = \frac{u_1 + u_2 + \cdots + u_n}{n}.$$

In case of an arithmetic progression, it follows from Eq. (10–3) that

$$\bar{u} = \frac{a+l}{2}.$$

This concept is useful in connection with partial payment plans where the time is less than a year so that the use of simple discount or simple interest would be appropriate.

EXAMPLE 10–3. A man borrows money from a bank which uses a simple discount rate of 8%. He promises to repay \$100 a month at the end of each month for the next 10 months. How much does he receive?

This transaction may be considered as ten simple transactions where \$100 each is borrowed for 1, 2, . . . , 10 months. The amount he receives, as computed on the basis of the simple discount formula (Section 4–5) is

$$\begin{aligned} P &= \$100[(1 - \tfrac{1}{12}8\%) + (1 - \tfrac{2}{12}8\%) + \cdots + (1 - \tfrac{10}{12}8\%)] \\ &= \$1000[1 - \tfrac{1}{10}(\tfrac{1}{12} + \tfrac{2}{12} + \cdots + \tfrac{10}{12})8\%] \\ &= \$1000[1 - \tfrac{55}{120}8\%] = \$1000[1 - \tfrac{11}{24}8\%] = \$963.33. \end{aligned}$$

The coefficient of 8% is the average time $\bar{t}$, expressed in years. Hence $P = \$1000(1 - d\bar{t})$, where d is the simple discount rate.

More generally, suppose that payments of R are at the ends of 1, 2, . . . , t months, and the annual simple discount rate is d. The present values of these payments are

$$P_1 = R\left(1 - \frac{d}{12}\right), \quad P_2 = R\left(1 - \frac{2d}{12}\right), \cdots, \quad P_t = R\left(1 - \frac{td}{12}\right)$$

and their sum is

$$\begin{aligned} P &= (tR)\left(1 - \frac{1 + 2 + 3 + \cdots + t}{12t}d\right) \\ &= A(1 - \bar{t}d), \end{aligned} \tag{10–4}$$

where A is total amount repaid and $\bar{t}$ is the average time expressed in years for which the equal sums are borrowed. In Example 10–3, $\bar{t}$ is readily recognized as $5\frac{1}{2}$ months or $\frac{11}{24}$ of a year, and the formula can be applied directly.

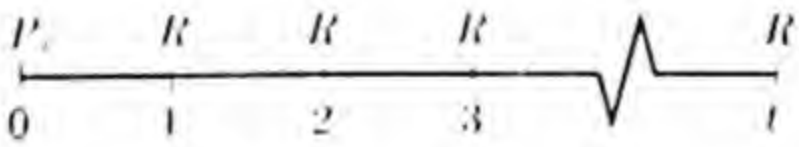


FIGURE 10–1

The equation of present value (Section 4–4) at the simple interest rate r_1, is

$$P = R\left\{\frac{1}{1 + \frac{r_1}{12}} + \frac{1}{1 + \frac{2r_1}{12}} + \frac{1}{1 + \frac{3r_1}{12}} + \cdots + \frac{1}{1 + \frac{tr_1}{12}}\right\}. \tag{10–5}$$

The sequence of terms within the braces do not form an arithmetic progression. If R, r_1, and t are given, P can be computed; but if P, R, and $t > 2$ are given, it would be difficult to determine the value of r_1. Since the due dates are different, there is no convenient comparison date for accumulated amounts when using simple interest. As suggested earlier, the use of compound interest would obviate this difficulty.

Corresponding to any simple discount rate there is a corresponding interest rate for the same time period (Section 4–5), such that

$$1 + \bar{t}r = \frac{1}{1 - \bar{t}d}.$$

When written in terms of this equivalent interest rate r, Eq. (10–4) becomes

$$A = \frac{P}{1 - \bar{t}d} = P(1 + \bar{t}r). \tag{10–6}$$

In terms of the amount of simple interest,

$$I = P\bar{t}r. \tag{10–7}$$

Partial payment plan problems solved by means of Eqs. (10–4) or (10–6) are said to be solved by the *average time* method.

EXAMPLE 10–4. A common plan used for automobile sales financing is to increase the unpaid balance of the cash price by 6% and divide this by 12 to obtain the monthly payment for each of 12 months. What is the corresponding simple interest rate, based on the average time method?

Suppose the unpaid balance were \$1200. The monthly payment would be \$1272/12 = \$106. The average time is $6\frac{1}{2}$ months or $\frac{13}{24}$ of a year. If Eq. (10–7) is used to find the rate r, then

$$\$72 = \$1200 \times \tfrac{13}{24} \times r,$$

so that

$$r = \frac{72 \times 24}{12 \times 13}\% = 11.08\%.$$

The corresponding simple discount rate as determined by Eq. (10–4) is 10.45%.

PROBLEM SET 10–1

1. The first four terms of a sequence are given below. Determine those sequences which could be arithmetic progressions. On the assumption that they are such progressions, find the twentieth term of the progression and also find the sum of the first twenty terms.

(a) 3, 7, 11, 15
(b) 2, 6, 10, 13
(c) $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}$
(d) $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}$
(e) $\frac{1}{3}, 1, 2, \frac{7}{3}$
(f) 34, 30, 26, 22

2. Determine c so that the following are the first three terms of arithmetic progressions. In each case find the twelfth term and the sum of the first twelve terms of the corresponding progressions.

(a) $-2, c, 8$
(b) $-2, 8, c$
(c) $\frac{3}{4}, c, \frac{3}{4}c$
(d) $\frac{4}{3}, c, \frac{2}{3}c^2$

3. Find the sum of all integers between 30 and 300 that are exactly divisible by 8.

4. Find the sum of all integers between 1 and 100 (inclusive) that are not divisible by 7.

5. Find the sum of the first n natural numbers by means of the formula for an arithmetic progression. Prove this formula correct by using mathematical induction.

6. Prove by mathematical induction that

(a) $1 + 2 + 3 + \cdots + (n - 1) = \dfrac{n(n - 1)}{2}, \qquad (n > 1);$

(b) $4 + 6 + 8 + \cdots + 2(n + 1) = n(n + 3).$

7. (a) Determine the sum of the first n positive integers of the form $3n - 1$, using the formula for the sum of an arithmetic progression. Verify the result by mathematical induction. (b) Repeat part (a) for integers of the form $3n + 1$.

8. Prove by mathematical induction that

$$2 + 2^2 + \cdots + 2^n = 2(2^n - 1).$$

9. Prove Eq. (10-2),

$$S_n = na + \frac{(n - 1)n}{2} d,$$

by mathematical induction.

10. If n is a positive integer, prove by mathematical induction that $x^{2n} - 1$ is divisible by $x + 1$.

11. A man borrows from a bank that uses a simple discount rate of 8%. He promises to repay \$50 at the end of each month for the next 12 months. How much does he receive? Use the average time method.

12. A man borrows \$560 net and promises to repay \$50 per month for 12 months. What are the corresponding simple discount rate and simple interest rate, based on the average time method?

13. Verify the simple discount rate given in Example 10–4.

14. A common plan used for automobile sales financing is to increase the unpaid balance by 9% and divide this by 18 to obtain the monthly payment for each of 18 months. What are the simple discount and simple interest rates based on the average time method?

15. One credit purchasing plan is to increase the cost by 10% and divide this by 6 to obtain the monthly payment for the next six months. What are the simple discount and simple interest rates based on the average time method?

10–3 Geometric progression. DEFINITION. A *geometric progression* is a sequence of numbers such that each number, after the first, is obtained from the previous one by multiplying a fixed number r.

If the first term is a, the sequence is

$$a, ar, ar^2, ar^3, \ldots, u_n = ar^{n-1}.$$

The form of u_n is checked for $n = 1, 2, 3$, and since

$$(ar^{n-1})r = ar^n = ar^{(n+1)-1},$$

it follows by mathematical induction that

$$u_n = ar^{n-1} \tag{10–8}$$

for any positive integer n.

The sum of the first n terms of the geometric progression ($n = 1, 2, 3, \ldots$) form a second sequence

$$S_1 = a, \qquad S_2 = a(1 + r), \qquad S_3 = a(1 + r + r^2), \qquad \ldots,$$
$$S_n = a(1 + r + r^2 + r^3 + \cdots + r^{n-1}).$$

This formula looks like, and is, the result of dividing $1 - r^n$ by $1 - r$. The proof that

$$S_n = a\frac{1 - r^n}{1 - r} \tag{10–9}$$

is given by mathematical induction. This is verified for $n = 1$ and $n = 2$, since

$$\frac{1 - r}{1 - r} = 1 \qquad \text{and} \qquad \frac{1 - r^2}{1 - r} = 1 + r.$$

Further,

$$S_{n+1} = a\frac{1 - r^n}{1 - r} + ar^n = a\left(\frac{1 - r^n}{1 - r} + \frac{r^n(1 - r)}{1 - r}\right)$$

$$= a\frac{1 - r^n + r^n - r^{n+1}}{1 - r} = a\frac{1 - r^{n+1}}{1 - r}.$$

It follows by induction that Eq. (10–9) is true for any positive integer n.

The same result can be obtained by forming rS and subtracting this from S.

If u_n is considered the last term and is represented by the symbol l, an equivalent formula is

$$S = \frac{a - rl}{1 - r} = \frac{rl - a}{r - 1}. \tag{10–10}$$

If $r > 1$, it is convenient to make the numerator and denominator in Eqs. (10–9) and (10–10) positive.

EXAMPLE 10–5. Find the sum of all integers between 1 and 10,000 that are powers of 3.

The first few such integers are 3, 9, 27, ... To find the last, without trial, solve the exponential equation $3^x = 10{,}000$ to the nearest integer:

$$x = \frac{\log 10{,}000}{\log 3} = \frac{4}{0.48} = 8^+.$$

$3^8 = (3^4)^2 = (81)^2 = 6561$. Hence

$$S = \frac{3(6561) - 3}{3 - 1} = 9840.$$

In this problem it was possible to find all the numbers and add them, but if the last number had been very large, this would have been laborious.

EXAMPLE 10–6. Prove by mathematical induction that the sum of the squares of the first n odd integers is $n(4n^2 - 1)/3$. Do these squares form an arithmetic or a geometric progression?

It is easy to verify that the formula is true for $n = 1$ and 2, since $1^2 = 3/3$ and $1^2 + 3^2 = 10 = 2(15)/3$. Assume

$$S_k = 1^2 + 3^2 + 5^2 + \cdots + (2k - 1)^2$$

$$= \frac{k(4k^2 - 1)}{3} = \frac{k(2k + 1)(2k - 1)}{3}.$$

Then

$$S_{k+1} = S_k + (2k + 1)^2 = \frac{k(2k + 1)(2k - 1)}{3} + \frac{3(2k + 1)^2}{3}$$

$$= \frac{2k + 1}{3}(2k^2 - k + 6k + 3) = \frac{2k + 1}{3}(2k^2 + 5k + 3)$$

$$= \frac{2k + 1}{3}(2k + 3)(k + 1) = \frac{(k + 1)(2k + 3)(2k + 1)}{3}$$

$$= \frac{(k + 1)(2[k + 1] + 1)(2[k + 1] - 1)}{3}.$$

which is of the form of S_k, except k has been replaced by $k + 1$. Hence the given formula is true for any positive integer n.

The sequence is neither an arithmetic nor a geometric progression.

If $u_1, u_2, \ldots, u_n$ is a sequence of positive numbers, its geometric average $\bar{g}$ is defined by

$$\bar{g} = \sqrt[n]{u_1 u_2 \ldots u_n}. \tag{10–11}$$

Hence

$$\log \bar{g} = \frac{\log u_1 + \log u_2 + \cdots + \log u_n}{n} \tag{10–12}$$

and this logarithmic form is used in numerical calculations. Note that $\log \bar{g}$ is the arithmetic average of the logarithms of the given u's. In the case of a geometric progression, it follows from Eq. (10–8) that

$$\begin{aligned} \bar{g} &= (a \times ar \times ar^2 \times \cdots \times ar^{n-1})^{1/n} \\ &= ar^{(1+2+3+\cdots+n-1)/n}. \end{aligned}$$

It can be shown by induction (See problem 6a, Problem Set 10–1) or by the formula (10–2) of arithmetic progressions, with n replaced by $(n - 1)$, that

$$[1 + 2 + 3 + \cdots + (n - 1)] = \frac{n(n - 1)}{2}, \qquad (n > 1).$$

Hence

$$\bar{g} = ar^{(n-1)/2} = \sqrt{al}. \tag{10–13}$$

This formula is valid for any two positive numbers.

EXAMPLE 10–7. Find the geometric average of (a) the first six multiples of 2; (b) the first six integral powers of 2.

(a) $\bar{g} = \sqrt[6]{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12} = \sqrt[6]{46{,}080}$

$\log \bar{g} = \frac{1}{6} \log 46{,}080 = \frac{1}{6}(4.6635) = 0.7772.$

$\bar{g} = 5.987.$

(b) The sequence is 2, 4, 8, 16, 32, 64.

Hence $\bar{g} = \sqrt{128} = 11.31.$

In a partial payment plan where compound interest is used, the result is independent of the comparison date. The terms form a geometric progression and hence can be summed by Eq. (10–10).

EXAMPLE 10–8 (compare Example 10–3). A man borrows money from a bank which uses a compound interest rate of 8%, compounded monthly. He promises to repay \$100 per month at the end of each month for the next 10 months. How much does he receive?

The amount he receives is the present value of ten payments of \$100 each.

$$P = \$100[(1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-10}],$$

where $i = \frac{2}{3}\%$. If Eq. (10–9) is used with $a = (1 + i)^{-1}$, $r = (1 + i)^{-1}$ and $n = 10$, the result is

$$P = \$100 \frac{(1 + i)^{-1} - (1 + i)^{-11}}{1 - (1 + i)^{-1}}.$$

If both numerator and denominator are multiplied by $1 + i$,

$$P = \$100 \frac{1 - (1 + i)^{-10}}{i},$$

where $i = \frac{2}{3}\%$. If it is given that $(1 + i)^{-10}$, $i = \frac{2}{3}\%$, is 0.93571 (see Example 10–11; this rate is not given in Table IV), then

$$P = \$10{,}000(1 - 0.93571) \times \tfrac{3}{2} = \$964.4.$$

More generally, if payments of R are made at the end of each month and the interest rate is i per month for n months, the equation of present value is

$$P = R(v + v^2 + v^3 + \cdots + v^n), \qquad v = (1 + i)^{-1},$$

$$= R \frac{v - v^{n+1}}{1 - v}$$

$$P = R \frac{1 - v^n}{i}.$$

If v^n is available in the table or can be computed by other methods, and if R is known, P can be computed. If P, R, and n are known, the problem of computing i is not difficult if appropriate tables are used. This will be discussed in detail in the section on annuities (Section 10–6).

Infinite geometric progression. If the geometric progression continues indefinitely and if $|r| < 1$, it is possible to compute a sum for it. For example, the decimal fraction equivalent of 1/3 is

$$0.333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{100} + \cdots \text{(indefinitely)},$$

which is a geometric progression with $a = 3/10$ and $r = 1/10$. Such a progression is called an *infinite geometric* progression. It has no last term, but the numerical value of u_n may become arbitrarily small in absolute value when n becomes very large. This occurs if and only if $|r| < 1$, in which case $|r^n|$ tends to zero as n increases without limit. In that case, the sum of the progression, indicated by S_∞, is

$$S_\infty = \frac{a}{1 - r}, \tag{10–14}$$

in the sense that $a(1 - r^n)/(1 - r)$ and $a/(1 - r)$ can be made to differ by as small a number as we please by choosing n sufficiently large.

EXAMPLE 10–9. A stick is 2 ft long. Half of it is cut off, then half of what is left is cut off, then half of what is left is cut off, and this process is continued indefinitely. How much of the stick is eventually cut off?

The answer is 2 ft in the sense that the amount cut off can be made as close to 2 ft as we please. In terms of an infinite geometric progression, the amount cut off is

$$S_\infty = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

EXAMPLE 10–10. A certain stock is expected to pay an annual dividend of \$1 at the end of every year. If it is assumed that money will always be worth 5%, what is the value of the stock dividends?

The equation of present value is

$$P = (1.05)^{-1} + (1.05)^{-2} + \cdots + (1.05)^{-n} + \cdots.$$

The sum of this infinite geometric progression is

$$P = \frac{(1.05)^{-1}}{1 - (1.05)^{-1}} = \frac{1}{0.05} = \$20.$$

Examination of the table of present values shows that the present value of the 50th payment is less than 9 cents, and the value of the 100th payment is less than 0.8 cents.

Any repeating decimal fraction may be considered the sum of a finite decimal fraction and an infinite geometric progression whose ratio is $1/10^k$, where k is the length of the period. Equation (10–14) shows that it is equivalent to a rational fraction and gives a means to determine this rational fraction. The repeating decimal fraction 0.3333 . . . has a first term of 3/10, a ratio of 1/10, and hence

$$0.3333\cdots = \frac{3/10}{1 - 1/10} = \frac{3}{9} = \frac{1}{3}.$$

EXAMPLE 10–11. Find the rational fraction which corresponds to the repeating decimal fraction $2.3\dot{7}0\dot{2}$.

$$N = \frac{23}{10} + \frac{702}{10000}\left(1 + \frac{1}{1000} + \frac{1}{1000^2} + \cdots\right).$$

The sum of the infinite geometric progression within the parentheses is

$$\frac{1}{1 - 0.001} = \frac{1000}{999}.$$

Hence

$$N = \frac{23}{10} + \frac{702}{10{,}000} \times \frac{1000}{999} = \frac{23}{10} + \frac{26}{370} = \frac{877}{370}.$$

10–4 Binomial Theorem. The expansions

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b + 4ab^3 + b^4$$

$$= a^4 + \frac{4}{1}a^3b + \frac{4 \cdot 3}{1 \cdot 2}a^2b^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}ab^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}b^4$$

are special cases of the Binomial Theorem. If n is a positive integer, then

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$$

$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n. \quad (10\text{–}15)$$

The special form

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)}{3!}x^3 + \cdots + x^n, \quad (10\text{–}16)$$

where $k!$ is the product of positive integers from 1 to k, is a useful form of the Binomial Theorem. Since $(a + b)^n = a^n(1 + b/a)^n = a^n(1 + x)^n$, where $x = b/a$, the theorem may be proved in the form of Eq. (10–16).

If it is assumed that Eq. (10–16) is true up to a certain n, $(1 + x)^{n+1}$ is formed by multiplying $(1 + x)^n$ by x and adding this to $(1 + x)^n$. The first few terms, after changing like powers to a common denominator, are

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$x(1 + x)^n = \quad x + \frac{n \cdot 2}{2}x^2 + \frac{n(n-1)}{2} \cdot \frac{3}{3}x^3 + \cdots$$

$$(1 + x)^{n+1} = 1 + (n + 1)x + \frac{n(n - 1 + 2)}{2!}x^2$$

$$+ \frac{n(n-1)(n - 2 + 3)}{3!}x^3 + \cdots$$

Reordering the factors yields

$$(1+x)^{n+1} = 1 + (n+1)x + \frac{(n+1)([n+1]-1)}{2!}x^2 + \frac{(n+1)([n+1]-1)([n+1]-2)}{3!}x^3 + \cdots,$$

which is of the form of Eq. (10–16) except that n has been replaced by $n+1$. The coefficient of x^k in the expansion of $(1+x)^{n+1}$ is

$$\frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)!} + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\cdot\frac{k+1}{k+1}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k+k+1)}{(k+1)!}$$

$$= \frac{(n+1)([n+1]-1)([n+1]-2)\cdots([n+1]-k)}{(k+1)!}.$$

It follows by induction that the Binomial Theorem is true for any positive integral value of n.

It is shown by means of the calculus* that Eq. (10–16), omitting the last term, is also true for fractional values and negative values of n, provided $|x| < 1$. This condition is satisfied by $(1+i)^n$, where i is any of the ordinary interest rates. In these cases, the sequence of terms continues indefinitely but only a few terms are required to find a good decimal approximation to $(1+i)^n$.

EXAMPLE 10–12. (a) Compute $\sqrt{1.06}$ to five decimal places using the Binomial Theorem and compare with the result obtained by the division process. (b) In Example 9–18, the value of $(1.06)^{-1/4}$ was required. Find this value by means of the Binomial Theorem. (c) In Example 10–8, the value of $(1.00\frac{2}{3})^{-10}$ was required. Find this value by means of the Binomial Theorem.

(a) $$(1.06)^{1/2} = 1 + \frac{1}{2}(0.06) + \frac{(1/2)(-1/2)(0.0036)}{2} + \frac{(1/2)(-1/2)(-3/2)}{6}(0.000216) + \cdots$$

$$= 1.030000 - 0.000450 + 0.000013$$

$$= 1.02956.$$

---

* The case where n is a positive integer is used to derive a fundamental formula of the calculus, and this is then used to extend the Binomial Theorem to the other cases.

The Binomial Theorem shows that the first approximation to $\sqrt{1.06}$ is 1.03:

$$\frac{1.06}{1.03} = 1.02912,$$

and the average of this and 1.03 is $1.02956 = \sqrt{1.06}$.

$$\begin{aligned}(b)\ (1.06)^{-1/4} &= 1 - \tfrac{1}{4}(0.06) + \tfrac{1}{2}(-\tfrac{1}{4})(-\tfrac{5}{4})(0.0036) \\ &\qquad + \tfrac{1}{6}(-\tfrac{1}{4})(-\tfrac{5}{4})(-\tfrac{9}{4})(0.000216) + \cdots \\ &= 1 - 0.015000 + 0.000563 - 0.000025 + \cdots \\ &= 0.98454.\end{aligned}$$

$$\begin{aligned}(c)\ (1.00\tfrac{2}{3})^{-10} &= 1 - 10(0.00\tfrac{2}{3}) + \tfrac{1}{2}(-10)(-11)(0.0000\tfrac{1}{9}) \\ &\qquad + \tfrac{1}{6}(-10)(-11)(-12)(0.000000\tfrac{8}{27}) + \cdots \\ &= 1.000000 - 0.066667 + 0.002442 - 0.000066 \\ &= 0.93571,\end{aligned}$$

which was the value previously used.

Problem Set 10–2

1. The first four terms of a sequence are given below. Determine those sequences which could be geometric progressions. On the assumption that they are geometric progressions, find the tenth term of each progression and the sum of the first ten terms.

(a) 16, 8, 4, 2
(b) $\frac{1}{4}$, $\frac{1}{2}$, 1, $\frac{3}{2}$
(c) $\frac{1}{2}$, $-\frac{3}{4}$, $\frac{15}{8}$, $-\frac{27}{16}$
(d) 9, −12, 16, $-\frac{64}{3}$

2. Determine c, so that the following are the first three terms of a geometric progression. If $|c| < 1$, find the sum of the corresponding infinite geometric progressions.

(a) 2, c, 8
(b) −2, 8, c
(c) $\frac{3}{4}$, c, $\frac{3}{4}c$
(d) 4/3, c, 2/(9c)

3. Find the sum of all integers between 1 and 200,000 that are powers of 2.

4. Show that there are only two integers between 10,000 and 100,000 that are powers of 3. How many such integers are there between 10,000 and 1,000,000?

5. Prove Eq. (10-9): $S = a(1 - r^n)/(1 - r)$ by forming rS and subtracting this from S.

6. A particle starting at the origin moves 9 units to the right, $\frac{2}{3}$ of this distance to the left, $\frac{2}{3}$ of this second distance to the right, $\frac{2}{3}$ of this third distance to the left, and so on *ad infinitum*. (a) How far does it travel, and (b) what position does it approach? (Show that the answer to (a) is the sum of an infinite geometric progression all of whose terms are positive, and the answer to (b) is the sum of an infinite geometric progression whose terms alternate in sign.)

7. An elastic ball, dropped from a height of 1 ft, rebounds $\frac{2}{3}$ of the distance it falls each time. How far does it ultimately travel. (Show that this distance is the sum of an infinite geometric progression except for the first term.)

8. Find the rational equivalents of the following repeating decimal fractions.

(a) $0.3\dot{8}\dot{1}$ (b) $2.\dot{8}3\dot{7}$ (c) $2.4\dot{7}\dot{1}$

9. Find the geometric average of the following sets.

(a) the first six odd integers
(b) the first five integral powers of 3
(c) the first ten integral powers of 2
(d) 3, 6, 8, 11, 13, 15

10. If a and b are two positive integers, use the fact that $(\sqrt{a} - \sqrt{b})^2 > 0$ to prove the fact that their arithmetic average is greater than their geometric average.

11. Use the Binomial Theorem to compute $\sqrt{1.05}$ to five decimal places and compare the result with that obtained by the division process.

12. Compute $(1.005)^{20}$ and $(1.005)^{-20}$ to five decimal places by the Binomial Theorem and check the results from $(1.005)^{-20} = 1/(1.005)^{20}$.

13. A certain stock is expected to pay an annual dividend of \$6 at the end of each year. If it is assumed that money will always be worth $4\frac{1}{2}\%$, what is the value of the stock dividends? What is the present value of the dividend due in 100 years?

14. A man borrows money from a bank which uses a compound interest rate of 1% per month. He promises to pay \$100 at the end of each month for 6 months. What does he receive? (Write the equation of present value, sum the geometric progression, and evaluate $(1.01)^{-6}$ by means of the Binomial Theorem.)

15. A man borrows \$800 net from a bank which uses a compound interest rate of 1% per month. What equal amounts should he repay at the end of each month for the next 6 months? (See the suggestions to problem 14.)

16. A man deposits \$60 in a savings bank at the end of each 6 months for 10 years. If the interest rate is 4%, compounded semiannually, how much is to his credit just after the twentieth payment? (Write the equation of value at the end of 10 years, sum the progression and evaluate $(1.02)^{20}$ by means of the Binomial Theorem.)

10–5 Annuity formulas. *An annuity is a sequence of equal payments made at equal intervals.* If these payments are considered to earn compound interest, their values at any date form a geometric progression. For simplicity it is assumed that interest is compounded as often as the payments are made, that is, the payment period and the interest period are the same. The following notation is used:

n = number of payment periods

R = amount of each payment, *paid at the end of the period* (rent)

i = compound interest rate per period

S_n = accumulated value of all payments at the time of the last payment

A_n = present value of all payments, *one period before* the first payment is made.

If no misunderstanding is likely to occur, the subscripts on S_n and A_n are dropped. In that case A must not be confused with the accumulated amount of a single payment as used in Section 9–7. $s_{\overline{n}|}$ (read "s angle n") and $a_{\overline{n}|}$ (read "a angle n") are standard symbols used for S_n and A_n if the periodic rent is 1. If it is desired to state the interest rate explicitly, $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are written ($s_{\overline{n}|}$ at i) and ($a_{\overline{n}|}$ at i), so that

$$S_n = Rs_{\overline{n}|} \text{ at } i; \qquad A_n = Ra_{\overline{n}|} \text{ at } i. \tag{10–17}$$

To find $s_{\overline{n}|}$ (Fig. 10–2), observe that the last payment is paid at the comparison date; the payment before that must be accumulated for 1 period, the payment before that for 2 periods, and so on; the first payment is accumulated for $n - 1$ periods. Hence

$$s_{\overline{n}|} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1},$$

which is a geometric progression with ratio $(1 + i)$ and with sum (Eq. (10–10))

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}. \tag{10–18}$$

Similarly, by finding the present value of each payment of 1 in the same order,

$$a_{\overline{n}|} = (1 + i)^{-n} + (1 + i)^{-n+1} + \cdots + (1 + i)^{-2} + (1 + i)^{-1}.$$

This geometric progression, with ratio $(1 + i)$, is summed by the formula used above to find

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}. \tag{10–19}$$

If the expanded forms of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are compared, it is seen that

$$a_{\overline{n}|} = (1 + i)^{-n} s_{\overline{n}|}. \tag{10–20}$$

Equation (10–19) could be obtained from Eq. (10–18). This shows one of

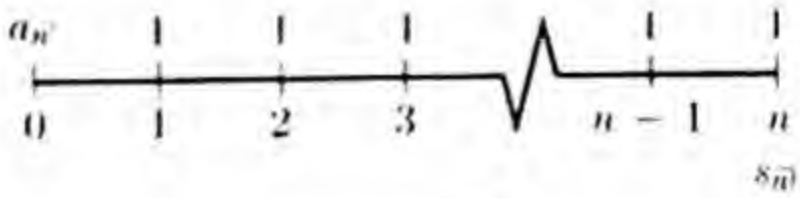


FIGURE 10–2

the advantages of using compound interest instead of simple interest in partial payment plans.

If the payments continue indefinitely, the annuity is called a *perpetuity* and its present value is given by

$$A_x = R\,\frac{1}{i}, \tag{10–21}$$

since $(1 + i)^{-n}$ approaches zero as n increases and is made sufficiently large.

EXAMPLE 10–13. An alumnus wishes to endow his university with a perpetual annual scholarship of \$300. If the funds can be invested over a period of years at an average rate of $4\frac{1}{4}\%$, how much must he donate to the university?

$$A = \$300\,\frac{1}{0.0425} = \$7060, \text{ approx.}$$

10–6 Annuity tables. Equations (10–17), (10–18), (10–19) show that if any three of the four quantities S (or A), R, n, i are given, the other is determined. The numerical calculations may be complicated, and special tables of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ have been prepared (Tables V and VI) for selected interest rates and selected value of n. The tabulated values are given to five decimal places, or to six significant figures, which is sufficient for most purposes. If the values of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are needed for some interest rate not included in the tables, the calculations can be made by means of the Binomial Theorem. If the rate i is included but n is not, $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are computed by methods discussed in the next section. In this section, and the corresponding problem set, only those i and n are used where $s_{\overline{n}|}$ and $a_{\overline{n}|}$ can be read directly from the tables. Equation (10–17) and the tables are used in the solution of the following problems.

(a) Given R, n, i, find S_n and A_n.

$$S_n = Rs_{\overline{n}|}, \qquad A_n = Ra_{\overline{n}|} \tag{10–17}$$

and $s_{\overline{n}|}$ and $a_{\overline{n}|}$ can be read directly from the tables.

(b) Given S_n or A_n, n, i, find R.

$s_{\overline{n}|}$ and $a_{\overline{n}|}$ can be read directly from the tables, so

$$R = \frac{S_n}{s_{\overline{n}|}} \qquad \text{and} \qquad R = \frac{A_n}{a_{\overline{n}|}}.$$

The values of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ usually contain more significant figures than are needed to compute R to the nearest cent. These should be rounded off so that they contain one more significant figure than is needed for R and the final result then rounded off to the nearest cent.

(c) Given S_n and R (a similar analysis holds if A_n and R are given), Eq. (10–17) is reduced to the simpler form $s_{\overline{n}|} = c$ by division. If n is also given, the value of i is determined by interpolation, first locating c between two entries in the table for different i's. The accuracy of such interpolation is ordinarily not more than 1/100 of 1%.

(d) If S_n, R, and i are given, the value of n is found from the table using $s_{\overline{n}|} = c$ at i. A fractional value of n has no apparent significance. If n is located between two successive integers, n_1 and $n_1 + 1$, then $n_1 + 1$ full payments give more than S_n and the problem is to determine the number of regular payments R and the size of an irregular payment X which would be paid at the end of $(n_1 + 1)$ periods. Similar remarks apply to the equation $a_{\overline{n}|} = k$ at i. Both problems are solved by using the appropriate equations of value.

Annuity problems are usually stated in words. It is part of the problem to write the corresponding equation. Care must be exercised to distinguish between a present value and the value at the end of the payment periods. The four problems mentioned above are now illustrated.

EXAMPLE 10–14. A man deposits \$30 at the end of each month with a building and loan association which pays interest at the rate of 6% compounded monthly. What does he have to his credit at the end of 10 years?

$$\begin{aligned} S &= \$30\, s_{\overline{120}|} \text{ at } \tfrac{1}{2}\% \\ &= \$30(163.879) = \$4916.37. \end{aligned}$$

EXAMPLE 10–15. A man borrows \$800 which he promises to repay in equal monthly installments over the next 20 months with interest at the rate of 7% compounded monthly. What is the monthly payment?

$$\$800 = Ra_{\overline{20}|} \text{ at } 7/12\%$$

$$R = \frac{800}{18.826} = \$42.49$$

In order to avoid excessive calculations, $a_{\overline{20}|}$ was rounded off to five significant figures. The last cent may be doubtful.

EXAMPLE 10–16. How long will it take to accumulate \$5000 by depositing \$100 at the end of each year in an account that earns $4\frac{1}{2}\%$ per year?

The equation of value, based on 1, is

$$s_{\overline{n}|} = 50 \text{ at } 4\tfrac{1}{2}\%.$$

The table shows that n lies between 26 and 27 and that if the full 27th

payment were made, the account would contain \$5071.13. Hence 26 full payments of \$100 are required and if the fund is to contain exactly \$5000, the last irregular payment is \$100 − 71.13 = \$28.87.

EXAMPLE 10–17. A man borrows \$1000 which he promises to repay at the rate of \$50 per month. If the interest rate is $\frac{1}{2}\%$ per month, how long will it take to repay the debt and what is the size of last irregular payment if this is made one month after the last regular payment?

The equation of present value is

$$1000 = 50a_{\overline{n}|} \text{ at } \tfrac{1}{2}\%$$

$$20 = a_{\overline{n}|} \text{ at } \tfrac{1}{2}\%.$$

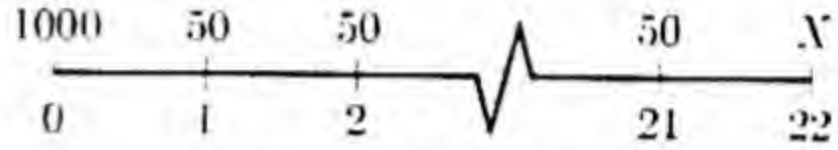


FIGURE 10–3

The table shows that n lies between 21 and 22 with $a_{\overline{21}|} = 19.8880$. If the irregular payment X is taken into account (see Fig. 10–3), the equation of value is

$$1000 = 50a_{\overline{21}|} + X(1.005)^{-22}$$

$$1000 - 994.40 = 5.60 = X(1.005)^{-22}$$

$$X = 5.60(1.005)^{22}$$

$$= 5.60(1.116) = \$6.25.$$

EXAMPLE 10–18. At what rate of interest would annual payments of \$1 for 25 years accumulate to \$40?

The equation of value is

$$s_{\overline{25}|} = 40 \text{ at } i.$$

The tables show that the rate lies between 4% and $3\frac{1}{2}\%$, and interpolation is done as follows:

	$s_{\overline{25}|}$		
$3\frac{1}{2}\%$	3895	105	
i	4000		270
4%	4165		

The tabular values are rounded off and the decimal point omitted to simplify the calculations:

$$i = \left(3\tfrac{1}{2} + \frac{105}{270} \times \frac{1}{2}\right)\% = 3.69\%.$$

Problem Set 10–3

1. Compute the accumulated value S_n and the present value A_n of annuities with the following data.

(a) $R = \$40$, $n = 35$, $i = 0.045$
(b) $R = \$25$, $n = 15$, $i = 6\%$
(c) $R = \$100$, $n = 120$, $i = \frac{1}{2}\%$
(d) $R = \$30$, $n = 70$, $i = 0.025$

2. Compute the periodic rent R for annuities with the following data.

(a) $S_n = \$2500$, $n = 50$, $i = 7\%$
(b) $A_n = \$1500$, $n = 25$, $i = 0.05$
(c) $S_n = \$400$, $n = 120$, $i = 1\%$
(d) $A_n = \$200$, $n = 25$, $i = 1\frac{3}{4}\%$

3. A man deposits \$75 at the end of each six months in a savings bank which pays interest at the rate of $3\frac{1}{2}\%$ compounded semiannually. How much is to his credit at the end of 8 years?

4. A man borrows \$2000 which he promises to repay with compound interest at the rate of $\frac{7}{12}\%$ per month, in equal monthly payments over the next 3 years. What is the monthly payment?

5. What fund set aside now and invested at 3% per year would provide annual payments of \$100 beginning 1 year from now and continuing for a period of (a) 10 years; (b) 100 years; (c) forever?

6. A man buys a house for \$4000 down and \$75 a month for the next 15 years. If the interest rate is 6% compounded monthly, what is the equivalent cash price of the house?

7. A man buys a house worth \$10,000. He pays \$4000 down and promises to pay the balance in monthly payments over a period of 15 years. If money is worth 5%, compounded monthly, what is the monthly rent?

8. A man wishes to collect at least \$4000 by depositing \$100 at the end of each half-year with a building and loan association which pays 5% compounded semiannually. How long will it take? After the last regular deposit, he permits the fund to accumulate for another half-year. At the end of this half-year, will he have to make a partial payment? Explain.

9. A man borrows \$2400, which he promises to repay at the rate of \$60 a month. If the interest rate is 1% a month, how long will it take to repay the debt? What irregular payment will be due one month after the last regular payment?

10. A man buys jewelry worth \$300 on a partial payment plan under which 12% of the price is added, and he promises to repay one-twelfth of this total (\$28) at the end of each month during the next 12 months. What is the corresponding interest rate, compounded monthly?

11. Under a so-called "18 month-9% plan," the cash value is increased by 9%, and one-eighteenth of this is to be paid at the end of each month for eighteen months. In the fine print on the agreement appears the statement: "This is not 9% interest." What is the equivalent yearly interest rate, compounded monthly?

12. If one-third of the price of a car has been paid in cash, the XYZ-bank offers to finance a loan of \$700 for the total repayment of \$748 over a period of 18 months. (a) What simple discount rate (average time method) does the bank earn? (b) What compound interest rate?

13. At what rate of interest would annual payments of \$1 for 30 years accumulate to \$60?

10–7 Special annuity problems. In this section a variety of problems related to the evaluation of annuities are considered.

(A) *Interest rate is not in the table.** In this case $s_{\overline{n}|}$ and $a_{\overline{n}|}$ are evaluated (using Eqs. (10–18) and (10–19)) by means of the Binomial Theorem. In Example 10–8 it was necessary to evaluate $a_{\overline{10}|}$ at $\frac{2}{3}\%$; $(1 + i)^{-10}$ was evaluated by means of the Binomial Theorem in Example 10–12c. If S_n or A_n are required, enough significant figures are needed (six or seven) to guarantee the desired accuracy. If the periodic rent is the unknown, such accuracy is not needed. If n is the unknown, this can be found for the equation

$$(1 + i)^n = is_{\overline{n}|} + 1,$$

using logarithms. It is actually simpler to use a process of double interpolation, since n must be an integer.

EXAMPLE 10–19. How long will it take to accumulate approximately \$4000 by depositing \$25 at the end of each quarter of a year in a bank that pays $4\frac{1}{2}\%$ compounded quarterly?

The periodic interest rate of $1\frac{1}{8}\%$ is not given in our table. By means of entries in the table of $s_{\overline{n}|}$ at rates 1% and $1\frac{1}{4}\%$, the corresponding equation of value

$$s_{\overline{n}|} \text{ at } 1\tfrac{1}{8}\% = 160$$

can be solved for n by double interpolation.

n	1%	Est. $1\frac{1}{8}\%$	$1\frac{1}{4}\%$
91	147.3	157.6	167.8
92	149.8	160.3	170.9
93	152.3	163.3	174.0

* Extensive tables are available in complete texts on the subject of the mathematics of finance or in separate publications. For example, see W. L. Hart, *Mathematics of Investment*, 3rd. ed., 1946, Heath; O. B. Tabor, *Mathematics of Finance*, 1952, Addison-Wesley; Dyess and Gilmore, *Mathematics of Business and Finance*, including tables by F. C. Kent and M. E. Kent, 1942, McGraw-Hill. These sources were consulted in preparing and checking the tables in this text.

The table shows that it will require 92 payments, approximately, that is, it will require 23 years. A more precise table shows that 92 payments are not quite enough.

EXAMPLE 10–20. A man borrows \$800, which he promises to repay in equal monthly installments over a period of 2 years with interest at $7\frac{1}{2}\%$ compounded monthly. What is the monthly payment?

The rate per period is $\frac{15}{24}\% = \frac{5}{8}\%$, a rate not given in our table. The equation of value is

$$\$800 = Ra_{\overline{24}|} \text{ at } \tfrac{5}{8}\% = R\,\frac{1 - (1+i)^{-24}}{i}$$

where $i = \frac{5}{8}\%$. To compute $a_{\overline{24}|}$ it is first necessary to compute $(1.00\frac{5}{8})^{-24}$. The Binomial Theorem gives

$$\begin{aligned}(1.00\tfrac{5}{8})^{-24} &= 1 - 24(0.00\tfrac{5}{8}) + \frac{24\cdot 25}{2}(0.00\tfrac{5}{8})^2 \\ &\quad - \frac{24\cdot 25\cdot 26}{2\cdot 3}(0.00\tfrac{5}{8})^3 + \frac{24\cdot 25\cdot 26\cdot 27}{2\cdot 3\cdot 4}(0.00\tfrac{5}{8})^4 \\ &\quad - \frac{24\cdot 25\cdot 26\cdot 27\cdot 28}{2\cdot 3\cdot 4\cdot 5}(0.00\tfrac{5}{8})^5 + \cdots \\ &= 1.000000 - 0.150000 + 0.011719 \\ &\quad - 0.000635 + 0.000026 - 0.000001 \\ &= 0.86111.\end{aligned}$$

Consequently,

$$a_{\overline{24}|} = \frac{1 - 0.86111}{0.00625} = \frac{0.13889}{0.00625} = 22.222.$$

Hence

$$R = \frac{\$800}{22.222} = \$36.00.$$

(B) *Coupon bonds.* In the simplest form a coupon bond is a promise to pay the redemption price F at a specified date and in the meantime to pay periodic dividends at regular intervals. (This type of bond differs from U.S. Government Savings Bonds, which merely promise to pay a fixed redemption value at a later date and are sold at a price below this redemption value.) The dividend D is the face value of the bond times the periodic dividend rate r. Such a bond may be purchased at a yield rate i which differs from the dividend rate r. The problem is to determine the price P of the bond n periods before the redemption date, given the face value F of the bond (as its redemption value), the dividend rate, and

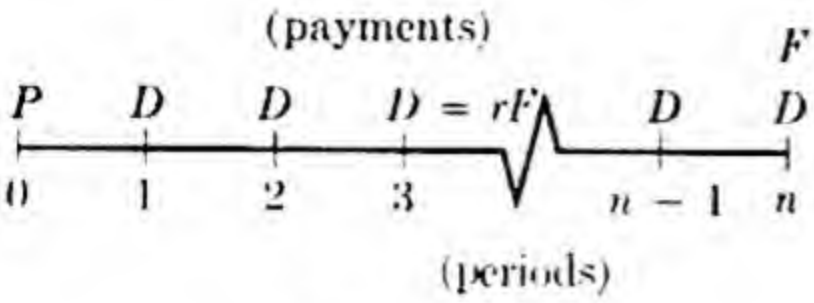


FIGURE 10–4

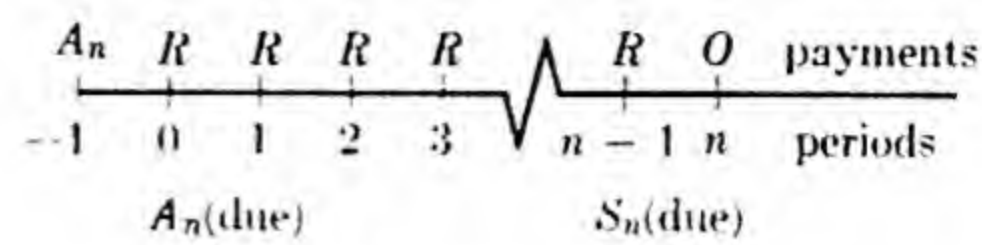


FIGURE 10–5

the yield rate. This price is the present value of F plus the present value of an annuity. (See Fig. 10–4.)

$$P = F(1 + i)^{-n} + rFa_{\overline{n}|} \text{ at } i, \tag{10–22}$$

and P is found by using Tables IV and VI.

EXAMPLE 10–21. What should be paid for a \$100 bond due in 10 years with dividend rate 5% compounded semiannually, to yield the purchaser (a) 4% compounded semiannually, (b) 6% compounded semiannually.

$$\text{(a) } P = \$100(1.02)^{-20} + 2.50a_{\overline{20}|} \text{ at } 2\%$$
$$= \$100(0.67297) + 2.50(16.3514) = \$108.18$$

$$\text{(b) } P = \$100(1.03)^{-20} + \$2.50a_{\overline{20}|} \text{ at } 3\%$$
$$= \$100(0.55368) + \$2.50(14.8775) = \$92.56.$$

The values of $a_{\overline{20}|}$ could be rounded off to five significant figures before multiplying.

(C) *Value of an annuity at any date.* So far the value of an annuity has been found at the beginning of the term or at its end. The equation of value could equally well be found at any date. Two methods, referred to as the *prospective method* and the *retrospective method*, are available and these may involve an additive process or a multiplicative process. These methods are discussed under several subheadings.

(a) *Annuity due.* An annuity due is an annuity whose payments are made at the beginning of the periods. (See Fig. 10–5.)

From the point of view of the prospective method, an annuity due is a standard annuity of $(n - 1)$ payments plus one cash payment. Looking backward, if the sequence were evaluated one period before the first payment, its value is the present value of a standard annuity. This value accumulated for one period gives the value at the time of the first payment. If $R = 1$, the corresponding formulas are

$$a_{\overline{n}|} \text{ (due)} = 1 + a_{\overline{n-1}|} = a_{\overline{n}|}(1 + i). \tag{10–23}$$

Similarly, if the value of the annuity is found when the last payment is made, its value one period later is found by accumulating it for one period. On the other hand, if it is supposed that an additional payment was made

at the end of the nth period the value at that date would be $s_{\overline{n+1}|}$. Since no payment was made then, indicated by the 0 in the diagram, its value is subtracted to find the value of an annuity due n periods after the first payment. Hence

$$s_{\overline{n}|}\,(\text{due}) = s_{\overline{n+1}|} - 1 = s_{\overline{n}|}(1 + i). \tag{10–24}$$

EXAMPLE 10–22. Compute the present value of an annuity due of \$40 per period for 10 periods and the value at the end of the 10th period, if the interest rate is $1\frac{3}{4}\%$ per period. Use both methods.

$$\begin{aligned} A_n\,(\text{due}) &= \$40(1 + a_{\overline{9}|}\text{ at }1\tfrac{3}{4}\%) \\ &= \$40(1 + 8.2605) = \$370.42 \end{aligned}$$

$$\begin{aligned} A_n\,(\text{due}) &= \$40a_{\overline{10}|}\,(1.0175) \\ &= \$40(9.1012)(1.0175) = \$40(9.2605) \\ &= \$370.42, \quad \text{as before} \end{aligned}$$

$$\begin{aligned} S_n\,(\text{due}) &= \$40(s_{\overline{11}|} - 1) \\ &= \$40(12.0148 - 1) = \$440.59 \end{aligned}$$

$$\begin{aligned} S_n\,(\text{due}) &= \$40(s_{\overline{10}|})(1.0175) \\ &= \$40(10.8254)(1.0175) \\ &= \$40(11.0148) = \$440.59, \quad \text{as before.} \end{aligned}$$

EXAMPLE 10–23. How long will it take to collect \$1000 by depositing \$10 at the beginning of each month in a savings account which pays interest at the rate of $\frac{1}{2}\%$ per month?

$$\begin{aligned} \$1000 &= 10s_{\overline{n}|}\,(\text{due}) \text{ at } \tfrac{1}{2}\% \\ &= 10(s_{\overline{n+1}|} - 1) \end{aligned}$$

$$s_{\overline{n+1}|} = 101 \text{ at } \tfrac{1}{2}\%.$$

The tables show that $s_{\overline{81}|} = 99.558$ and $s_{\overline{82}|} = 101.056$. Hence $n + 1 = 82$ and $n = 81$. If 81 deposits are made, the last at the beginning of the 81st month, the accumulated amount at the time of the 81st deposit is \$995.58. At the end of the month, the interest of \$4.98 is added which makes the amount \$1000.56. No additional deposit is needed.

(b) *Deferred annuity.* An annuity is said to be deferred for k periods if the first payment is made at the end of the $(k + 1)$st period (Fig. 10–6). If there are n payments, the present value of these payments of 1 each can be found in two ways. Looking backward, the value of these n payments at the end of the kth period is $a_{\overline{n}|}$; and if this is discounted to the

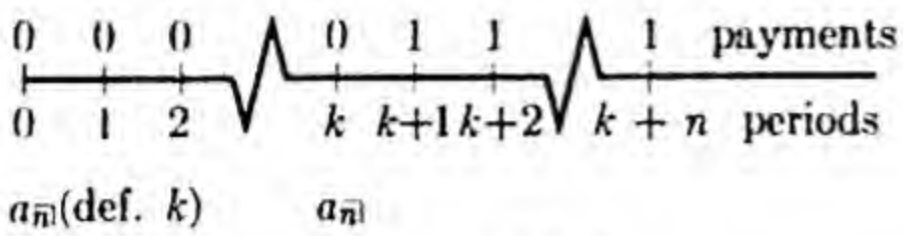


FIGURE 10–6

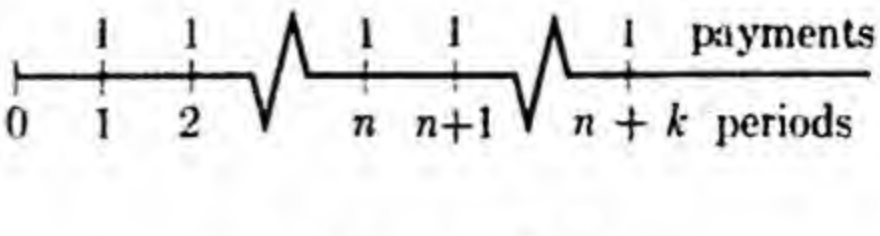


FIGURE 10–7

present, $a_{\overline{n}|}$ (def. k) is obtained. On the other hand if it is supposed that payments were made during the first k periods, the present value of the $(n + k)$ payments would have been $a_{\overline{n+k}|}$. Since these k payments were not made, as indicated by the zeros, the present value of these k payments must be subtracted to obtain $a_{\overline{n}|}$ (def. k). Thus

$$a_{\overline{n}|}\,(\text{def. } k) = a_{\overline{n+k}|} - a_{\overline{k}|} = a_{\overline{n}|}(1 + i)^{-k}. \tag{10–25}$$

In a similar manner formulas could be derived which give the value of an annuity of n payments k periods after the last payment is made, that is, an annuity that is forwarded k periods. If this value is represented by $s_{\overline{n}|}$ (for. k),

$$s_{\overline{n}|}\,(\text{for. } k) = s_{\overline{n}|}(1 + i)^{k} = s_{\overline{n+k}|} - s_{\overline{k}|}. \tag{10–26}$$

The notion of a deferred annuity is used for finding the value of an annuity when the number of periods is not in the table. Suppose that the number of periods is $n + k$, where n and k do correspond to entries in the table. The required annuity may be considered as the sum of an annuity for n periods plus an annuity for k periods which is deferred n periods (Fig. 10–7). Hence

$$a_{\overline{n+k}|} = a_{\overline{n}|} + a_{\overline{k}|}(1 + i)^{-n}. \tag{10–27}$$

$$s_{\overline{n+k}|} = s_{\overline{n}|}(1 + i)^{k} + s_{\overline{k}|}. \tag{10–28}$$

A similar analysis can be applied to the case of an annuity when the rent changes after n periods. Suppose that for the first n periods the rent is R and for the next k periods the rent is R'. Then

$$A = Ra_{\overline{n}|} + R'a_{\overline{k}|}(1 + i)^{-n}. \tag{10–29}$$

$$S = Rs_{\overline{n}|}(1 + i)^{k} + R's_{\overline{k}|}. \tag{10–30}$$

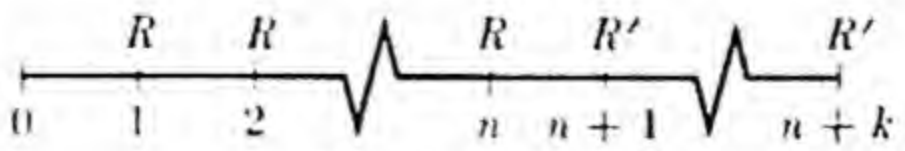


FIGURE 10–8

If $R' > R$, an alternative procedure for Eq. (10–29) would be to consider $R = R' - (R' - R)$, so that

$$A = R'a_{\overline{n+k}|} - (R' - R)a_{\overline{n}|}. \tag{10–29′}$$

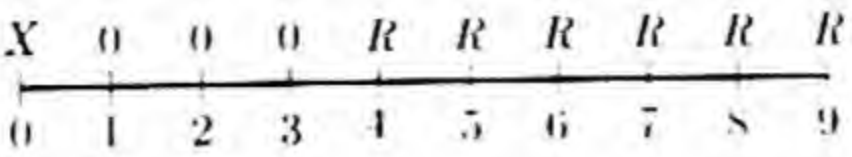


FIGURE 10–9

EXAMPLE 10–24. Miss Adams, needing cash to go on a long vacation trip, borrows money from a loan company which uses the interest rate of 12%, compounded monthly. She promises to repay the loan in 6 equal payments of \$100 each, the first to be made 4 months after the loan is made. How much cash does she receive? (See Fig. 10–9.)

$$X = \$100a_{\overline{6}|} \text{ (def. 3)} = \$100(a_{\overline{9}|} - a_{\overline{3}|}) \text{ at } 1\%.$$
$$= \$100(8.5660 - 2.9410) = \$562.50$$

or

$$X = \$100a_{\overline{6}|}(1.01)^{-3}.$$

EXAMPLE 10–25. Find an approximate value of $a_{\overline{94}|}$ and of $s_{\overline{94}|}$ at $\frac{1}{2}\%$.

$$a_{\overline{94}|} = a_{\overline{90}|} + a_{\overline{4}|}(1.005)^{-90}$$
$$= 72.3313 + 3.95050(0.63834)$$
$$= 72.3313 + 2.5218 = 74.8531.$$

Because $(1.005)^{-90}$ was known only to five significant figures, one decimal place is lost.

$$s_{\overline{94}|} = s_{\overline{90}|}(1.005)^4 + s_{\overline{4}|}$$
$$= (113.3109)(1.02015) + 4.03010$$
$$= 115.5941 + 4.0301 = 119.624.$$

Because only six significant figures were known for $(1.005)^4$, the final result is rounded off to six significant figures.

EXAMPLE 10–26. If money is worth 5% a year, what is the present value of the set of payments consisting of (1) \$100 at the end of each year for the next 10 years and \$200 at the end of each year for the following 5 years; (2) \$200 at the end of each year for five years and \$100 a year for the following 10 years.

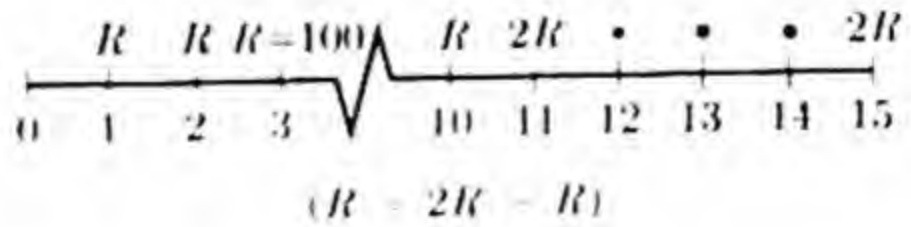


FIGURE 10–10

$$X = \$100a_{\overline{10}|} + 200a_{\overline{5}|}(1.05)^{-10} = \$1303.77.$$

Considering $R = 2R - R$, the calculation is simpler from the form

$$X = 200a_{\overline{15}|} - 100a_{\overline{10}|}$$
$$= 200(10.37966) - 100(7.72173) = 1303.76.$$

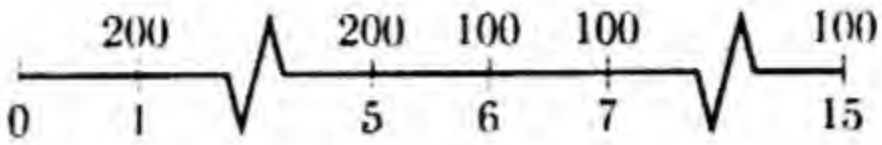


FIGURE 10–11

$$Y = \$100a_{\overline{13}|} + 100a_{\overline{5}|}$$
$$= \$100(10.37966 + 4.32948) = \$1470.91.$$

(c) *Amount remaining due.* If the first k payments in a partial payment plan have been made, the amount remaining due, A_{n-k}, immediately after the kth payment has been made can be found as follows. If n is an integer, so is $n - k$ and if $R = 1$, the required value is $a_{\overline{n-k}|}$. The periodic rent may be determined by an equation of the type

$$A = Ra_{\overline{n}|}, \qquad \text{or} \qquad R = \frac{A}{a_{\overline{n}|}},$$

in which case

$$A_{n-k} = Ra_{\overline{n-k}|} = \frac{Aa_{\overline{n-k}|}}{a_{\overline{n}|}}. \tag{10-31}$$

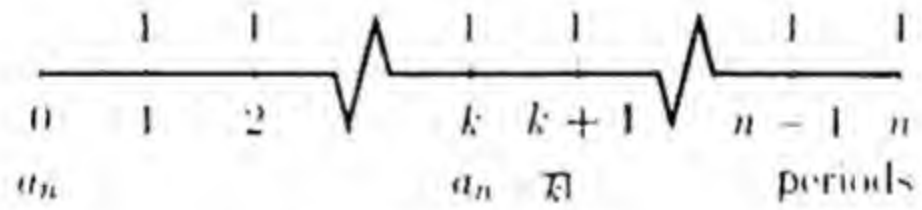


FIGURE 10–12

The computation may be long, but it is necessary to carry enough significant figures in $a_{\overline{n}|}$ to make the division give a correct answer.

In partial payment plans, where A, R and i are specified, the number n would not in general be an integer, so there may be N regular payments and an irregular payment one period after the last regular one. The first problem to solve is to find the size of the irregular payment. It is a special case of the "amount remaining due" problem after N payments have been made. The general problem can be solved in a manner which does not differ greatly from this special problem.

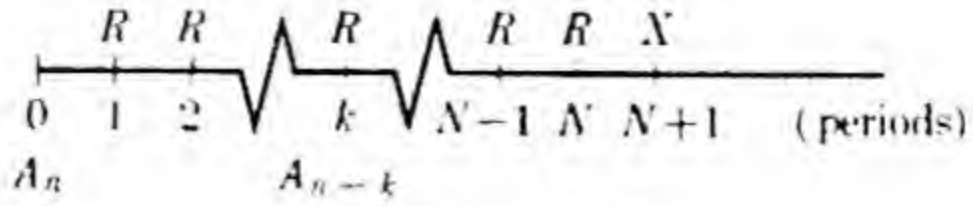


FIGURE 10–13

First suppose N has been found from the equation of value $A_n = Ra_{\overline{n}|}$, through use of a table. Then the equation of value can be written

$$A_n = Ra_{\overline{N}|} + X(1 + i)^{-(N+1)}.$$

If this is solved for X,

$$X = (A_n - Ra_{\overline{N}|})(1 + i)^{N+1}. \tag{10-32}$$

Another method is to write the equation of value at the end of N periods and then solve for X:

$$A_n(1 + i)^N = Rs_{\overline{N}|} + X(1 + i)^{-1}$$

$$X = [A_n(1 + i)^N - Rs_{\overline{N}|}](1 + i). \tag{10-33}$$

This is equivalent to finding the amount remaining due at the end of N periods and accumulating it for one period. Equations (10–32) and (10–33) are equivalent. If A_{n-k} represents the amount remaining due immediately after the kth payment, it may be found as the difference of the accumulated amount of A_n, as if no payments had been made, and the value at that date of the payments that have been made:

$$A_{n-k} = A_n(1 + i)^k - Rs_{\overline{k}|}. \tag{10-34}$$

EXAMPLE 10–27. Mr. Jones buys a house valued at \$20,000 and pays \$4000 down. He agrees to pay the balance with interest at 5% in 20 equal annual installments. What is the annual payment and how much does he still owe immediately after the tenth annual payment?

The equation of value to determine the size of the annual payment is

$$\$16{,}000 = Ra_{\overline{20}|} \text{ at } 5\%$$

$$R = \frac{16{,}000}{a_{\overline{20}|}} = \$1283.88.$$

Immediately after the tenth payment, 10 payments are still due and their total value is then $Ra_{\overline{10}|}$. Hence

$$\text{Amount due} = \frac{16{,}000a_{\overline{10}|}}{a_{\overline{20}|}} = 1283.88a_{\overline{10}|}$$

$$= 1283.88(7.72173) = 9913.77.$$

Although he has made 50% of the payments, he still owes 62% of the debt.

EXAMPLE 10–28. Mr. Smith buys a house valued at \$20,000 and pays \$4000 down. He agrees to pay the balance with interest at 5% by paying

$1200 at the end of each year until the debt is paid. (a) How many regular payments must be made and what is the size of the final irregular payment made one year after the last regular payment? (b) How much does he still owe immediately after half of the annual payments are made?

(a) The equation of value to determine the number of payments is

$$16{,}000 = 1200a_{\overline{n}|} \text{ at } 5\%$$

$$13.33 = a_{\overline{n}|} \text{ at } 5\%.$$

The table shows that 22 regular payments are needed. The present value of the amount remaining due immediately after the 22nd payment is

$$16{,}000 - 1200a_{\overline{22}|} = 16{,}000 - 1200(13.16300)$$
$$= \$204.40.$$

Hence the irregular payment, paid at the end of the twenty-third year, is

$$X = 204.40(1.05)^{23} = 204.40(3.07152)$$
$$= \$627.82.$$

(b) The amount remaining due immediately after the 11th payment is (Eq. 10-29)

$$\text{Amount due} = 16{,}000(1.05)^{11} - 1200s_{\overline{11}|}$$
$$= 16{,}000(1.71034) - 1200(14.2068)$$
$$= 27336.5 - 17048.2 = \$10{,}288.3.$$

Problem Set 10–4

1. What should be paid for a $100 bond due in 20 years, with dividend rate 6%, compounded semiannually, to yield the purchaser (a) 5% compounded semiannually, (b) 7% compounded semiannually?

2. Mr. Brown borrows $1000 and promises to repay it at the end of 12 years and, in the meantime, to pay interest on the loan at the end of each year at the rate of 5% per year. His promissory note is immediately sold to yield the purchaser $4\frac{1}{2}\%$ per year. What is the purchase price?

3. (a) Compute the value of an annuity due of $60 per period for 20 periods, if the interest rate is $1\frac{1}{2}\%$ per period. Use both methods. (b) Find the value of the same annuity at the end of the twentieth period. Use both methods.

4. A sequence of payments of $20 each are made for 10 years, the first at the end of the third (beginning of the fourth) year. Write the equations of value for (a) the beginning of the third year, (b) the end of the thirteenth year, (c) the present time, (d) the end of the twentieth year, and (e) the end of the eighth year. Include a line diagram but do not make the computations.

5. Mr. White deposits \$10 a month for 6 years in an account that pays interest at the rate of 6% compounded monthly. He then waits until the account accumulates to \$1000. How long does he wait?

6. By means of a line diagram and an appropriate explanation, justify Eq. (10–26):

$$s_{\overline{n}|}\,(\text{for. } k) = s_{n|}(1+i)^k = (s_{\overline{n+k}|} - s_{\overline{k}|}) \text{ at } i.$$

7. Verify that the second method suggested for Example 10–24 gives the same answer as the first method.

8. Miss Baker, needing cash to go on a long vacation trip, borrows money from a bank, which uses the interest rate of $\frac{3}{4}\%$ a month. She promises to repay the loan in 8 equal monthly installments of \$80 each, the first to be made 4 months after the loan is made. How much cash does she receive?

9. Write equations of value for determining $a_{\overline{n}|}$ and $s_{\overline{n}|}$ for the indicated n and i from values to be found in the tables. (This can be done in many ways.) Do not compute these values.

(a) $n = 240, \quad i = \frac{1}{2}\%$
(b) $n = 72, \quad i = 3\%$
(c) $n = 100, \quad i = 6\%$

10. Compute the values of $s_{\overline{51}|}$ and $a_{\overline{51}|}$ at 6% from values found in the tables.

11. If money is worth 5% a year, what is the value at the time of the last payment of the set of payments consisting of (a) \$100 at the end of each year for the next 10 years and \$200 at the end of each year for the following five years; (b) \$200 at the end of each year for five years and \$100 a year for the following 10 years? If possible, write more than one equation of value and use the simpler one for numerical calculations.

12. Mr. Green buys a second-hand car and promises to pay \$500 down and \$50 a month at the end of each month for 10 months and \$75 a month for the following 8 months. If the interest rate is $\frac{7}{12}\%$ a month, what is the equivalent cash price of the car?

13. Mr. Robins plans to pay off an indebtedness of \$1000 by making equal payments at the end of each month for the next two years. If the interest rate is 5% compounded monthly, (a) what is the monthly payment and (b) what does he still owe just after the first year?

14. Mr. Jones plans to pay off a trust deed of \$12,000 in 20 equal annual payments. If the interest rate is $4\frac{1}{2}\%$, (a) what is the annual payment and (b) how much does he still owe immediately after the twelfth payment?

15. Mr. Roberts plans to pay off an indebtedness of \$1000 by paying \$50 at the end of each month until the debt is paid. If the interest rate is 5% compounded monthly, (a) how many regular payments must he make, and (b) what is the size of the irregular payment made one month after the last regular payment? (c) How much does he still owe immediately after the twelfth payment?

16. Mr. Smith plans to pay off a trust deed of \$12,000 with interest at $4\frac{1}{2}\%$ by making annual payments of \$1000. (a) How many regular payments must he make, and (b) what is the size of the final regular payment made one year

after the last regular payment? (c) How much does he still owe immediately after the tenth payment?

17. How long will it take to accumulate approximately \$5000 by depositing \$40 at the end of each quarter of a year in a bank that pays $4\frac{1}{2}\%$ compounded quarterly? (The rate $1\frac{1}{8}\%$ is not in the tables.)

18. How long will it take to accumulate at least \$2500 by saving \$100 a year if all funds are invested at $5\frac{1}{2}\%$ per year? (The rate $5\frac{1}{2}\%$ is not in the tables.)

19. Verify the calculations made for $(1 + i)^{-24}$, $i = \frac{5}{8}\%$, in Example 10–20.

20. Compute $(1 + i)^{24}$, $i = \frac{5}{8}\%$ and check the result in Example 10–20, using $(1 + i)^{-n} = 1/(1 + i)^n$.

APPENDIX I

THE EQUATION OF A PLANE

A plane is completely characterized geometrically by the following property. If F and N are different points which determine a line, then the locus of points P such that $\overline{PF}$ is perpendicular to $\overline{FN}$ is a plane (Fig. I–1). These lines will be perpendicular if and only if

$$PN^2 = PF^2 + FN^2.$$

It is shown first that if $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two given points, then the distance P_1P_2 is given by

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Construct the rectangular box which has $\overline{P_1P_2}$ as its diagonal and whose faces are parallel to the coordinate planes. The planes parallel to the $x = 0$ plane determine two points L_1 and L_2 on the x-axis (see Fig. 5-6) with coordinates $L_1(x_1, 0, 0)$, $L_2(x_2, 0, 0)$. $L_1L_2 = |x_2 - x_1| = P_1Q$, where P_1Q is the length of one side of the box (Fig. I-2). In a similar fashion, the other edges of the box are found to be $QR = |y_2 - y_1|$ and $RP_2 = |z_2 - z_1|$. Hence, using solid geometry:

$$(P_1P_2)^2 = (P_1R)^2 + (z_2 - z_1)^2,$$

so that

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

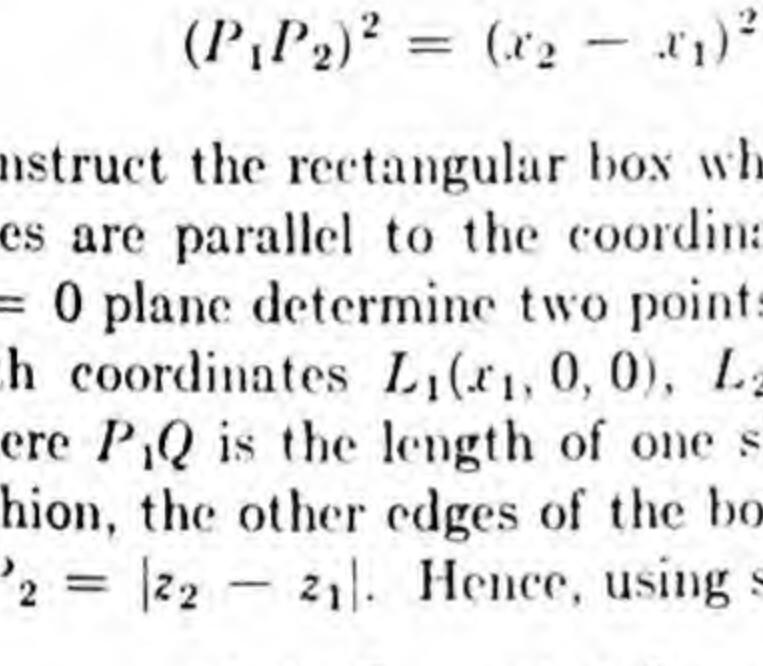


FIGURE I-1

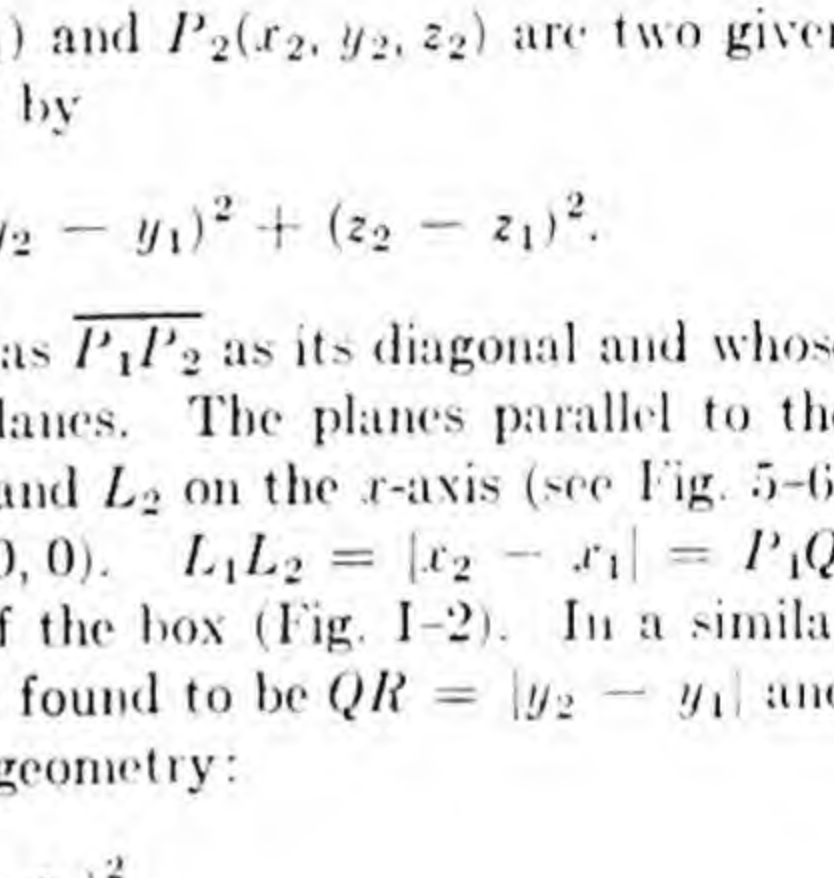


FIGURE I-2

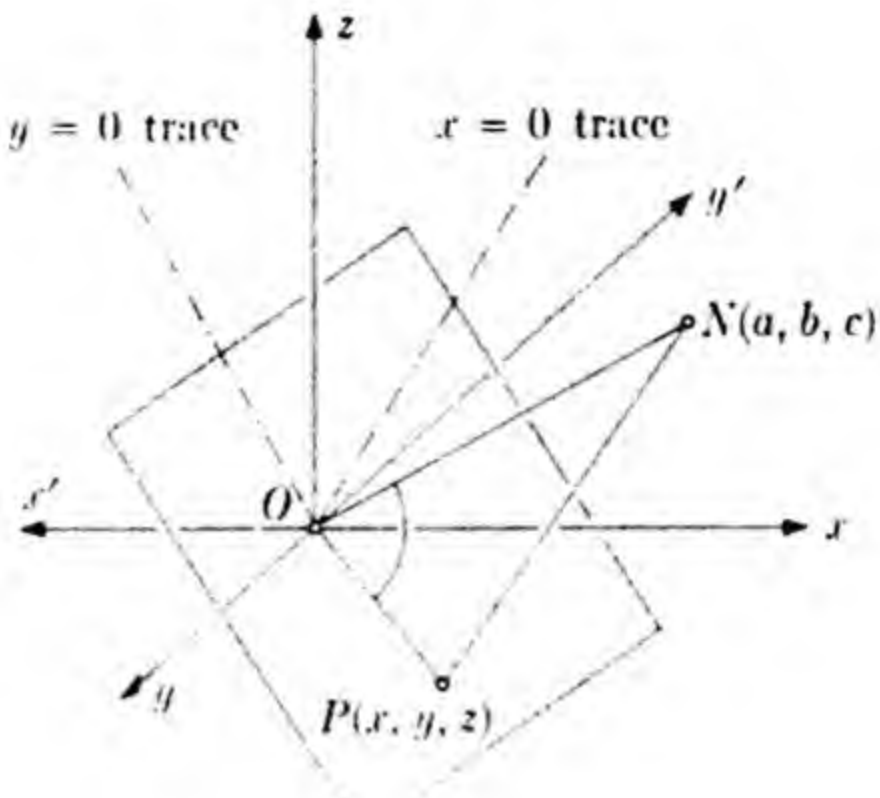


FIGURE I-3

We derive the equation of a plane in the two cases where (a) the plane passes through the origin (Fig. I-3); (b) the plane does not pass through the origin (Fig. I-4).

(a) If the plane passes through the origin, let $N(a, b, c)$ be any point on the perpendicular to the plane at the point O. Let $P(x, y, z)$ be any point in the plane. The equation $PN^2 = OP^2 + ON^2$, in terms of coordinates, becomes

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = (x^2 + y^2 + z^2) + (a^2 + b^2 + c^2).$$

This reduces to

$$ax + by + cz = 0,$$

and the steps are reversible. Hence this is the equation of the given plane, where (a, b, c) are the coordinates of a point on the line through O perpendicular to the plane. Fig. I-3 represents the plane $3x + y + 2z = 0$, whose normal ON lies in the first octant, but whose traces do not lie in the first quadrants of the coordinate planes.

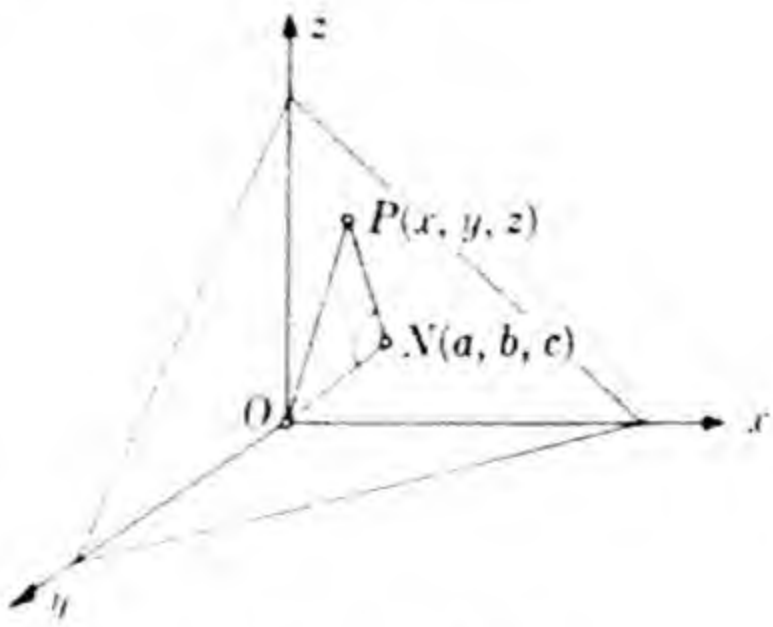


FIGURE I-4

(b) If the plane does not pass through O, let the perpendicular from O meet the plane in $N(a, b, c)$. Any other point on this normal, ON has coordinates of the form (ka, kb, kc). The equation $OP^2 = ON^2 + NP^2$ becomes

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + (x - a)^2 + (y - b)^2 + (z - c)^2$$

which reduces to

$$ax + by + cz = a^2 + b^2 + c^2,$$

and the steps are reversible. This is the equation of the given plane, where (a, b, c) are the coordinates of the foot of the perpendicular from O to the plane. For an equation of the form

$$ax + by + cz = d,$$

where $d \neq a^2 + b^2 + c^2$, a constant k can be determined so that the sum of squares of the coefficients of the equation obtained by multiplying through by k does equal to kd:

$$(ka)^2 + (kb)^2 + (kc)^2 = kd$$

gives $k = d/(a^2 + b^2 + c^2)$. For example, the plane $3x + y + 2z = 5$ can be reduced to required form by multiplying by $5/14$:

$$\tfrac{15}{14}x + \tfrac{5}{14}y + \tfrac{10}{14}z = \tfrac{25}{14},$$

where

$$(\tfrac{15}{14})^2 + (\tfrac{5}{14})^2 + (\tfrac{10}{14})^2 = \tfrac{25}{14}.$$

The point (a, b, c) always lies on the perpendicular drawn from the origin to the plane $ax + by + cz = d$. It follows that two planes

$$ax + by + cz = d_1 \quad \text{and} \quad ax + by + cz = d_2, \quad (d_1 \neq d_2),$$

are both perpendicular to the line joining $O(0, 0, 0)$ to $N(a, b, c)$ and hence are parallel. Such cases are thus easy to recognize algebraically. (See Section 5–9.)

The analysis above has been based upon the properties of Euclidean space with a universal unit of measure. At times in graphical representation, it is not convenient to select the same units along all axes. A change of units is equivalent to multiplying the coordinates of each point by constants:

$$k_1x' = x, \quad k_2y' = y, \quad k_3z' = z,$$

where (x, y, z) are the coordinates with a common unit of measure. The

equation of the plane becomes

$$ak_1x' + bk_2y' + ck_3z' = d,$$

which has the standard form:

$$a'x' + b'y' + cz' = d',$$

although now the formulas for distance and perpendicularity no longer apply. For graphical purposes, these formulas are not needed. Points, lines, and planes can be represented with the selected units taken into account and their intersections found as before.

APPENDIX II

THE DIVISION PROCESS FOR $\sqrt{n}$

Let x_1 be an approximation to $\sqrt{n}$ which is greater than $\sqrt{n}$. Then

$$x_1 = \sqrt{n} + p, \tag{II-1}$$

where p is a positive number such that $p < x_1$. Then

$$n = x_1^2 - 2x_1p + p^2$$

$$q_1 = \frac{n}{x_1} = x_1 - 2p + \frac{p^2}{x_1},$$

$$x_2 = \tfrac{1}{2}(x_1 + q_1) = x_1 - p + \frac{p^2}{2x_1}$$

$$x_2 = \sqrt{n} + \frac{p^2}{2x_1}. \tag{II-2}$$

This form shows that $x_2 > \sqrt{n}$. Since $p < x_1$, it follows that

$$\frac{p}{x_1} < 1 \qquad \text{and} \qquad \frac{p^2}{2x_1} < \frac{p}{2}.$$

Hence the difference between x_2 and $\sqrt{n}$ is less than $\frac{1}{2}$ the difference between x_1 and $\sqrt{n}$:

$$x_2 - \sqrt{n} < \frac{p}{2}. \tag{II-3}$$

If the division process is continued,

$$x_3 - \sqrt{n} < \frac{1}{2}\left(\frac{p}{2}\right) = \frac{p}{4},$$

and after carrying out the process m times to obtain x_{m+1},

$$x_{m+1} - \sqrt{n} < \frac{p}{2^m}, \tag{II-4}$$

which approaches zero as m increases. Hence $\sqrt{n}$ can be obtained to any desired number of significant figures.

The above discussion shows the convergence of the process to $\sqrt{n}$, but in actual practice, the convergence is more rapid than Eq. (II–4) indicates. Any process of finding $\sqrt{n}$ can be reduced to that of finding the square root of a number between 1 and 100, if the decimal points are properly adjusted. Without loss of generality, then, it is assumed that $1 < n < 100$,

so that $1 < \sqrt{n} < 10$. When x and q agree to two significant figures, then since $\sqrt{n}$ lies between them, x and $\sqrt{n}$ agree in the integral part and differ by at most 1 in the first decimal place. Hence $|p| = |x - \sqrt{n}| < 1/10$. When x and q agree to three significant figures, then $|p| < 1/100$. If x and q agree to k significant figures, then $|p| < 1/10^{k-1}$. Once the division process has reached the point that x and q agree to two significant figures, it makes little difference whether x is less than or greater than $\sqrt{n}$. In Eq. (II–1), $|p| < 1/10$, whereas in Eq. (II–2), the next error is

$$p^2/(2x) < 1/(200x).$$

There is agreement to at least two decimal places and hence to three (and probably more) significant figures. More generally, if $p < 1/10^{k-1}$, the next error is

$$\frac{p^2}{2x} < \frac{1}{2 \cdot 10^{2k-2} x},$$

which shows agreement to at least $(2k - 2)$ decimal places and at least $(2k - 1)$ significant figures. This justifies the rule stated in the text: *If x_1 and q_1 agree to k significant figures, carry the quotient to $2k$ significant figures and take their arithmetic average for $\sqrt{n}$. The last figure may be doubtful and the answer may be rounded off to one less figure.*

EXAMPLE. If the first approximation to $\sqrt{43}$ is $x_1 = 1$, then $q_1 = 43$ and $x_2 = 22$, which is not a better approximation than $x_1 = 1$, but is a better approximation than 43 and is larger than $\sqrt{43}$. The next approximation would be $x_3 = \frac{1}{2}(22 + 43/22) = 12$ approximately. The difference between 12 and $\sqrt{43}$ is less than $\frac{1}{2}$ the difference between 22 and $\sqrt{43}$. If the process is continued, the successive approximations become $x_4 = \frac{1}{2}(12 + 43/12) = 7$, $x_5 = \frac{1}{2}(7 + 43/7) = 6.6$. The next quotient is $q_5 = 6.5^+$, showing that x_5 and $\sqrt{43}$ differ by at most 1 in the second significant figure. Hence the division is continued to four significant figures to find $q_5 = 6.515$, $x_6 = \frac{1}{2}(6.600 + 6.515) = 6.557^+$. In the next step, $q_6 = 6.5578771$, carried to eight significant figures and $\sqrt{43} = 6.5574385$, which is correct to eight significant figures. The last digit can be trusted here because $2x_6 > 10$, but in other examples, round-off errors and accumulated errors make the last digit doubtful.

APPENDIX III

TABLES

TABLE I

SQUARES AND SQUARE ROOTS

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
1	1	1.000	3.162	51	2,601	7.141	22.58
2	4	1.414	4.472	52	2,704	7.211	22.80
3	9	1.732	5.477	53	2,809	7.280	23.02
4	16	2.000	6.325	54	2,916	7.348	23.24
5	25	2.236	7.071	55	3,025	7.416	23.45
6	36	2.449	7.746	56	3,136	7.483	23.66
7	49	2.646	8.367	57	3,249	7.550	23.87
8	64	2.828	8.944	58	3,364	7.616	24.08
9	81	3.000	9.487	59	3,481	7.681	24.29
10	100	3.162	10.00	60	3,600	7.746	24.49
11	121	3.317	10.49	61	3,721	7.810	24.70
12	144	3.464	10.95	62	3,844	7.874	24.90
13	169	3.606	11.40	63	3,969	7.937	25.10
14	196	3.742	11.83	64	4,096	8.000	25.30
15	225	3.873	12.25	65	4,225	8.062	25.50
16	256	4.000	12.65	66	4,356	8.124	25.69
17	289	4.123	13.04	67	4,489	8.185	25.88
18	324	4.243	13.42	68	4,624	8.246	26.08
19	361	4.359	13.78	69	4,761	8.307	26.27
20	400	4.472	14.14	70	4,900	8.367	26.46
21	441	4.583	14.49	71	5,041	8.426	26.65
22	484	4.690	14.83	72	5,184	8.485	26.83
23	529	4.796	15.17	73	5,329	8.544	27.02
24	576	4.899	15.49	74	5,476	8.602	27.20
25	625	5.000	15.81	75	5,625	8.660	27.39
26	676	5.099	16.12	76	5,776	8.718	27.57
27	729	5.196	16.43	77	5,929	8.775	27.75
28	784	5.292	16.73	78	6,084	8.832	27.93
29	841	5.385	17.03	79	6,241	8.888	28.11
30	900	5.477	17.32	80	6,400	8.944	28.28
31	961	5.568	17.61	81	6,561	9.000	28.46
32	1,024	5.657	17.89	82	6,724	9.055	28.64
33	1,089	5.745	18.17	83	6,889	9.110	28.81
34	1,156	5.831	18.44	84	7,056	9.165	28.98
35	1,225	5.916	18.71	85	7,225	9.220	29.15
36	1,296	6.000	18.97	86	7,396	9.274	29.33
37	1,369	6.083	19.24	87	7,569	9.327	29.50
38	1,444	6.164	19.49	88	7,744	9.381	29.66
39	1,521	6.246	19.75	89	7,921	9.434	29.83
40	1,600	6.325	20.00	90	8,100	9.487	30.00
41	1,681	6.403	20.25	91	8,281	9.539	30.17
42	1,764	6.481	20.49	92	8,464	9.592	30.33
43	1,849	6.557	20.74	93	8,649	9.644	30.50
44	1,936	6.633	20.98	94	8,836	9.695	30.66
45	2,025	6.708	21.21	95	9,025	9.747	30.82
46	2,116	6.782	21.45	96	9,216	9.798	30.98
47	2,209	6.856	21.68	97	9,409	9.849	31.14
48	2,304	6.928	21.91	98	9,604	9.899	31.30
49	2,401	7.000	22.14	99	9,801	9.950	31.46
50	2,500	7.071	22.36	100	10,000	10.000	31.62
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

TABLE II

COMMON LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
0	....	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4208
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7097	7016	7024	7033	7042	7050	7059	7067
N	0	1	2	3	4	5	6	7	8	9

TABLE II

COMMON LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7691	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9001	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9271	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039
N	0	1	2	3	4	5	6	7	8	9

TABLE III

ACCUMULATED AMOUNT OF 1
$(1 + i)^n$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
1	1.00333	1.00417	1.00500	1.00583	1.00750	1.01000	1
2	1.00668	1.00835	1.01002	1.01170	1.01506	1.02010	2
3	1.01003	1.01255	1.01508	1.01760	1.02267	1.03030	3
4	1.01340	1.01677	1.02015	1.02354	1.03034	1.04060	4
5	1.01678	1.02101	1.02525	1.02951	1.03807	1.05101	5
6	1.02017	1.02526	1.03038	1.03551	1.04585	1.06152	6
7	1.02357	1.02953	1.03553	1.04155	1.05370	1.07214	7
8	1.02698	1.03382	1.04071	1.04763	1.06160	1.08286	8
9	1.03040	1.03813	1.04591	1.05374	1.06956	1.09369	9
10	1.03384	1.04246	1.05114	1.05989	1.07758	1.10462	10
11	1.03728	1.04680	1.05640	1.06607	1.08566	1.11567	11
12	1.04074	1.05116	1.06168	1.07229	1.09381	1.12683	12
13	1.04421	1.05554	1.06699	1.07855	1.10201	1.13809	13
14	1.04769	1.05994	1.07232	1.08484	1.11028	1.14947	14
15	1.05118	1.06436	1.07768	1.09116	1.11860	1.16097	15
16	1.05469	1.06879	1.08307	1.09753	1.12699	1.17258	16
17	1.05820	1.07324	1.08849	1.10393	1.13544	1.18430	17
18	1.06173	1.07772	1.09393	1.11037	1.14396	1.19615	18
19	1.06527	1.08221	1.09940	1.11685	1.15254	1.20811	19
20	1.06882	1.08672	1.10490	1.12336	1.16118	1.22019	20
21	1.07238	1.09124	1.11042	1.12992	1.16989	1.23239	21
22	1.07596	1.09579	1.11597	1.13651	1.17867	1.24472	22
23	1.07954	1.10036	1.12155	1.14314	1.18751	1.25716	23
24	1.08314	1.10494	1.12716	1.14981	1.19641	1.26973	24
25	1.08675	1.10955	1.13280	1.15651	1.20539	1.28243	25
26	1.09038	1.11417	1.13846	1.16326	1.21443	1.29526	26
27	1.09401	1.11881	1.14415	1.17005	1.22354	1.30821	27
28	1.09766	1.12347	1.14987	1.17687	1.23271	1.32129	28
29	1.10132	1.12815	1.15562	1.18374	1.24196	1.33450	29
30	1.10499	1.13285	1.16140	1.19064	1.25127	1.34785	30
31	1.10867	1.13757	1.16721	1.19759	1.26066	1.36133	31
32	1.11237	1.14231	1.17304	1.20457	1.27011	1.37494	32
33	1.11607	1.14707	1.17891	1.21160	1.27964	1.38869	33
34	1.11979	1.15185	1.18480	1.21867	1.28923	1.40258	34
35	1.12353	1.15665	1.19073	1.22578	1.29890	1.41660	35
36	1.12727	1.16147	1.19668	1.23293	1.30865	1.43077	36
37	1.13103	1.16631	1.20266	1.24012	1.31846	1.44508	37
38	1.13480	1.17117	1.20868	1.24735	1.32835	1.45953	38
39	1.13858	1.17605	1.21472	1.25463	1.33831	1.47412	39
40	1.14238	1.18095	1.22079	1.26195	1.34835	1.48886	40
41	1.14619	1.18587	1.22690	1.26931	1.35846	1.50375	41
42	1.15001	1.19081	1.23303	1.27671	1.36865	1.51879	42
43	1.15384	1.19577	1.23920	1.28416	1.37891	1.53398	43
44	1.15769	1.20076	1.24539	1.29165	1.38926	1.54932	44
45	1.16154	1.20576	1.25162	1.29919	1.39968	1.56481	45
46	1.16542	1.21078	1.25788	1.30676	1.41017	1.58046	46
47	1.16930	1.21583	1.26417	1.31439	1.42075	1.59626	47
48	1.17320	1.22090	1.27049	1.32205	1.43141	1.61223	48
49	1.17711	1.22598	1.27684	1.32977	1.44214	1.62835	49
50	1.18103	1.23109	1.28323	1.33752	1.45296	1.64463	50
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE III

ACCUMULATED AMOUNT OF 1
$(1 + i)^n$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
51	1.18497	1.23622	1.28964	1.34533	1.46385	1.66108	51
52	1.18892	1.24137	1.29609	1.35317	1.47483	1.67769	52
53	1.19288	1.24654	1.30257	1.36107	1.48589	1.69447	53
54	1.19686	1.25174	1.30908	1.36901	1.49704	1.71141	54
55	1.20085	1.25695	1.31563	1.37699	1.50827	1.72852	55
56	1.20485	1.26219	1.32221	1.38502	1.51958	1.74581	56
57	1.20887	1.26745	1.32882	1.39310	1.53098	1.76327	57
58	1.21290	1.27273	1.33546	1.40123	1.54246	1.78090	58
59	1.21694	1.27803	1.34214	1.40940	1.55403	1.79871	59
60	1.22100	1.28336	1.34885	1.41763	1.56568	1.81670	60
61	1.22507	1.28871	1.35559	1.42589	1.57742	1.83486	61
62	1.22915	1.29408	1.36237	1.43421	1.58925	1.85321	62
63	1.23325	1.29947	1.36918	1.44258	1.60117	1.87174	63
64	1.23736	1.30488	1.37603	1.45099	1.61318	1.89046	64
65	1.24148	1.31032	1.38291	1.45946	1.62528	1.90937	65
66	1.24562	1.31578	1.38982	1.46797	1.63747	1.92846	66
67	1.24977	1.32126	1.39677	1.47653	1.64975	1.94774	67
68	1.25394	1.32677	1.40376	1.48515	1.66213	1.96722	68
69	1.25812	1.33229	1.41078	1.49381	1.67459	1.98689	69
70	1.26231	1.33785	1.41783	1.50252	1.68715	2.00676	70
71	1.26652	1.34342	1.42492	1.51129	1.69980	2.02683	71
72	1.27074	1.34902	1.43204	1.52011	1.71255	2.04710	72
73	1.27498	1.35464	1.43920	1.52897	1.72540	2.06757	73
74	1.27923	1.36028	1.44640	1.53789	1.73834	2.08825	74
75	1.28349	1.36595	1.45363	1.54686	1.75137	2.10913	75
76	1.28777	1.37164	1.46090	1.55589	1.76451	2.13022	76
77	1.29206	1.37736	1.46821	1.56496	1.77774	2.15152	77
78	1.29637	1.38310	1.47555	1.57409	1.79108	2.17304	78
79	1.30069	1.38886	1.48292	1.58327	1.80451	2.19477	79
80	1.30503	1.39465	1.49034	1.59251	1.81804	2.21672	80
81	1.30938	1.40046	1.49779	1.60180	1.83168	2.23888	81
82	1.31374	1.40629	1.50528	1.61114	1.84542	2.26127	82
83	1.31812	1.41215	1.51281	1.62054	1.85926	2.28388	83
84	1.32251	1.41804	1.52037	1.62999	1.87320	2.30672	84
85	1.32692	1.42394	1.52797	1.63950	1.88725	2.32979	85
86	1.33135	1.42988	1.53561	1.64907	1.90141	2.35309	86
87	1.33578	1.43584	1.54329	1.65869	1.91567	2.37662	87
88	1.34024	1.44182	1.55101	1.66836	1.93003	2.40038	88
89	1.34470	1.44783	1.55876	1.67809	1.94451	2.42439	89
90	1.34919	1.45386	1.56655	1.68788	1.95909	2.44863	90
96	1.37640	1.49059	1.61414	1.74783	2.04892	2.59927	96
100	1.39484	1.51558	1.64667	1.78897	2.11108	2.70481	100
108	1.43247	1.56585	1.71370	1.87418	2.24112	2.92893	108
120	1.49083	1.64701	1.81940	2.00966	2.45136	3.30039	120
132	1.55157	1.73127	1.93161	2.15494	2.68131	3.71896	132
144	1.61478	1.81985	2.05075	2.31072	2.93284	4.19062	144
156	1.68057	1.91296	2.17724	2.47776	3.20796	4.72209	156
168	1.74904	2.01083	2.31152	2.65688	3.50889	5.32097	168
180	1.82030	2.11370	2.45409	2.84895	3.83804	5.99580	180
192	1.89446	2.22185	2.60546	3.05490	4.19808	6.75622	192
200	1.94558	2.29700	2.71152	3.20040	4.45667	7.31602	200
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE III

ACCUMULATED AMOUNT OF 1
$(1 + i)^n$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.01250	1.01500	1.01750	1.02000	1.02500	1.03000	1
2	1.02516	1.03022	1.03531	1.04040	1.05062	1.06090	2
3	1.03797	1.04568	1.05342	1.06121	1.07689	1.09273	3
4	1.05095	1.06136	1.07186	1.08243	1.10381	1.12551	4
5	1.06408	1.07728	1.09062	1.10408	1.13141	1.15927	5
6	1.70738	1.09344	1.10970	1.12616	1.15969	1.19405	6
7	1.09085	1.10984	1.12912	1.14869	1.18869	1.22987	7
9	1.10449	1.12649	1.14888	1.17166	1.21840	1.26677	8
9	1.11829	1.14339	1.16899	1.19509	1.24886	1.30477	9
10	1.13227	1.16054	1.18944	1.21899	1.28008	1.34392	10
11	1.14642	1.17795	1.21026	1.24337	1.31209	1.38423	11
12	1.16075	1.19562	1.23144	1.26824	1.34489	1.42576	12
13	1.17526	1.21355	1.25299	1.29361	1.37851	1.46853	13
14	1.18995	1.23176	1.27492	1.31948	1.41297	1.51259	14
15	1.20483	1.25023	1.29723	1.34587	1.44830	1.55797	15
16	1.21989	1.26899	1.31993	1.37279	1.48451	1.60471	16
17	1.23514	1.28802	1.34303	1.40024	1.52162	1.65285	17
18	1.25058	1.30734	1.36653	1.42825	1.55966	1.70243	18
19	1.26621	1.32695	1.39045	1.45681	1.59865	1.75351	19
20	1.28204	1.34686	1.41478	1.48595	1.63862	1.80611	20
21	1.29806	1.36706	1.43954	1.51567	1.67958	1.86029	21
22	1.31429	1.38756	1.46473	1.54598	1.72157	1.91610	22
23	1.33072	1.40838	1.49036	1.57690	1.76461	1.97359	23
24	1.34735	1.42950	1.51644	1.60844	1.80873	2.03279	24
25	1.36419	1.45095	1.54298	1.64061	1.85394	2.09378	25
26	1.38125	1.47271	1.56998	1.67342	1.90029	2.15659	26
27	1.39851	1.49480	1.59746	1.70689	1.94780	2.22129	27
28	1.41599	1.51722	1.62541	1.74102	1.99650	2.28793	28
29	1.43369	1.53998	1.65386	1.77584	2.04641	2.35657	29
30	1.45161	1.56308	1.68280	1.81136	2.09757	2.42726	30
31	1.46976	1.58653	1.71225	1.84759	2.15001	2.50008	31
32	1.48813	1.61032	1.74221	1.88454	2.20376	2.57508	32
33	1.50673	1.63448	1.77270	1.92223	2.25885	2.65234	33
34	1.52557	1.65900	1.80372	1.96068	2.31532	2.73191	34
35	1.54464	1.68388	1.83529	1.99989	2.37321	2.81386	35
36	1.56394	1.70914	1.86741	2.03989	2.43254	2.89828	36
37	1.58349	1.73478	1.90009	2.08069	2.49335	2.98523	37
38	1.60329	1.76080	1.93334	2.12230	2.55568	3.07478	38
39	1.62333	1.78721	1.96717	2.16474	2.61957	3.16703	39
40	1.64362	1.81402	2.00160	2.20804	2.68506	3.26204	40
44	1.72735	1.92533	2.14543	2.39005	2.96381	3.67145	44
48	1.81535	2.04348	2.29960	2.58707	3.27149	4.13225	48
50	1.86102	2.10524	2.38079	2.69159	3.43711	4.38391	50
52	1.90784	2.16887	2.46485	2.80033	3.61111	4.65089	52
56	2.00503	2.30196	2.64197	3.03117	3.98599	5.23461	56
60	2.10718	2.44322	2.83182	3.28103	4.39979	5.89160	60
64	2.21453	2.59314	3.03531	3.55149	4.85654	6.63105	64
68	2.32735	2.75227	3.25342	3.84425	5.36072	7.46331	68
72	2.44592	2.92116	3.48721	4.16114	5.91723	8.40002	72
76	2.57053	3.10041	3.73780	4.50415	6.53151	9.45429	76
80	2.70148	3.29066	4.00639	4.87544	7.20957	10.64089	80
n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n

TABLE III

ACCUMULATED AMOUNT OF 1
$(1 + i)^n$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n
1	1.03500	1.04000	1.04500	1.05000	1.06000	1.07000	1
2	1.07122	1.08160	1.09202	1.10250	1.12360	1.14490	2
3	1.10872	1.12486	1.14117	1.15762	1.19102	1.22504	3
4	1.14752	1.16986	1.19252	1.21551	1.26248	1.31080	4
5	1.18769	1.21665	1.24618	1.27628	1.33823	1.40255	5
6	1.22926	1.26532	1.30226	1.34010	1.41852	1.50073	6
7	1.27228	1.31593	1.36086	1.40710	1.50363	1.60578	7
8	1.31681	1.36857	1.42210	1.47746	1.59385	1.71819	8
9	1.36290	1.42331	1.48610	1.55133	1.68948	1.83846	9
10	1.41060	1.48024	1.55297	1.62889	1.79085	1.96715	10
11	1.45997	1.53945	1.62285	1.71034	1.89830	2.10485	11
12	1.51107	1.60103	1.69588	1.79586	2.01220	2.25219	12
13	1.56396	1.66507	1.77220	1.88565	2.13293	2.40985	13
14	1.61869	1.73168	1.85194	1.97993	2.26090	2.57853	14
15	1.67535	1.80094	1.93528	2.07893	2.39656	2.75903	15
16	1.73399	1.87298	2.02237	2.18287	2.54035	2.95216	16
17	1.79468	1.94790	2.11338	2.29202	2.69277	3.15882	17
18	1.85749	2.02582	2.20848	2.40662	2.85434	3.37993	18
19	1.92250	2.10685	2.30786	2.52695	3.02560	3.61653	19
20	1.98979	2.19112	2.41171	2.65330	3.20714	3.86968	20
21	2.05943	2.27877	2.52024	2.78596	3.39956	4.14056	21
22	2.13151	2.36992	2.63365	2.92526	3.60354	4.43040	22
23	2.20611	2.46472	2.75217	3.07152	3.81975	4.74053	23
24	2.28333	2.56330	2.87601	3.22510	4.04893	5.07237	24
25	2.36324	2.66584	3.00543	3.38635	4.29187	5.42743	25
26	2.44596	2.77247	3.14068	3.55567	4.54938	5.80735	26
27	2.53157	2.88337	3.28201	3.73346	4.82235	6.21387	27
28	2.62017	2.99870	3.42970	3.92013	5.11169	6.64884	28
29	2.71188	3.11865	3.58404	4.11614	5.41839	7.11426	29
30	2.80679	3.24340	3.74532	4.32194	5.74349	7.61226	30
31	2.90503	3.37313	3.91386	4.53804	6.08810	8.14511	31
32	3.00671	3.50806	4.08998	4.76494	6.45339	8.71527	32
33	3.11194	3.64838	4.27403	5.00319	6.84059	9.32534	33
34	3.22086	3.79432	4.46636	5.25335	7.25103	9.97811	34
35	3.33359	3.94609	4.66735	5.51602	7.68609	10.67658	35
36	3.45027	4.10393	4.87738	5.79182	8.14725	11.42394	36
37	3.57103	4.26809	5.09686	6.08141	8.63609	12.22362	37
38	3.69601	4.43881	5.32622	6.38548	9.15425	13.07927	38
39	3.82537	4.61637	5.56590	6.70475	9.70351	13.99482	39
40	3.95926	4.80102	5.81636	7.03999	10.28572	14.97446	40
41	4.09783	4.99306	6.07810	7.39199	10.90286	16.02267	41
42	4.24126	5.19278	6.35162	7.76159	11.55703	17.14426	42
43	4.38970	5.40050	6.63744	8.14967	12.25045	18.34435	43
44	4.54334	5.61652	6.93612	8.55715	12.98548	19.62846	44
45	4.70236	5.84118	7.24825	8.98501	13.76461	21.00245	45
46	4.86694	6.07482	7.57442	9.43426	14.59049	22.47262	46
47	5.03728	6.31782	7.91527	9.90597	15.46592	24.04571	47
48	5.21359	6.57053	8.27146	10.40127	16.39387	25.72891	48
49	5.39606	6.83335	8.64367	10.92133	17.37750	27.52993	49
50	5.58493	7.10668	9.03264	11.46740	18.42015	29.45703	50
n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n

TABLE IV

PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
1	0.99668	0.99585	0.99502	0.99420	0.99256	0.99010	1
2	0.99337	0.99172	0.99007	0.98843	0.98517	0.98030	2
3	0.99007	0.98760	0.98515	0.98270	0.97783	0.97059	3
4	0.98678	0.98351	0.98025	0.97700	0.97055	0.96098	4
5	0.98350	0.97942	0.97537	0.97134	0.96333	0.95147	5
6	0.98023	0.97536	0.97052	0.96570	0.95616	0.94205	6
7	0.97697	0.97131	0.96569	0.96010	0.94904	0.93272	7
8	0.97373	0.96728	0.96089	0.95453	0.94198	0.92348	8
9	0.97049	0.96327	0.95610	0.94900	0.93496	0.91434	9
10	0.96727	0.95927	0.95135	0.94350	0.92800	0.90529	10
11	0.96406	0.95529	0.94661	0.93802	0.92109	0.89632	11
12	0.96085	0.95133	0.94191	0.93258	0.91424	0.88745	12
13	0.95766	0.94738	0.93722	0.92717	0.90743	0.87866	13
14	0.95448	0.94345	0.93256	0.92180	0.90068	0.86996	14
15	0.95131	0.93954	0.92792	0.91645	0.89397	0.86135	15
16	0.94815	0.93564	0.92330	0.91114	0.88732	0.85282	16
17	0.94500	0.93175	0.91871	0.90585	0.88071	0.84438	17
18	0.94186	0.92789	0.91414	0.90060	0.87416	0.83602	18
19	0.93873	0.92404	0.90959	0.89538	0.86765	0.82774	19
20	0.93561	0.92020	0.90506	0.89018	0.86119	0.81954	20
21	0.93250	0.91639	0.90056	0.88502	0.85478	0.81143	21
22	0.92940	0.91258	0.89608	0.87989	0.84842	0.80340	22
23	0.92632	0.90880	0.89162	0.87479	0.84210	0.79544	23
24	0.92324	0.90503	0.88719	0.86971	0.83583	0.78757	24
25	0.92017	0.90127	0.88277	0.86467	0.82961	0.77977	25
26	0.91711	0.89753	0.87838	0.85965	0.82343	0.77205	26
27	0.91407	0.89381	0.87401	0.85467	0.81730	0.76440	27
28	0.91103	0.89010	0.86966	0.84971	0.81122	0.75684	28
29	0.90800	0.88640	0.86533	0.84478	0.80518	0.74934	29
30	0.90499	0.88273	0.86103	0.83988	0.79919	0.74192	30
31	0.90198	0.87906	0.85675	0.83501	0.79324	0.73458	31
32	0.89898	0.87542	0.85248	0.83017	0.78733	0.72730	32
33	0.89600	0.87178	0.84824	0.82536	0.78147	0.72010	33
34	0.89302	0.86817	0.84402	0.82057	0.77565	0.71297	34
35	0.89005	0.86456	0.83982	0.81581	0.76988	0.70591	35
36	0.88710	0.86098	0.83564	0.81108	0.76415	0.69892	36
37	0.88415	0.85740	0.83149	0.80638	0.75846	0.69200	37
38	0.88121	0.85385	0.82735	0.80170	0.75281	0.68515	38
39	0.87829	0.85030	0.82323	0.79705	0.74721	0.67837	39
40	0.87537	0.84677	0.81914	0.79243	0.74165	0.67165	40
41	0.87246	0.84326	0.81506	0.78783	0.73613	0.66500	41
42	0.86956	0.83976	0.81101	0.78326	0.73065	0.65842	42
43	0.86667	0.83628	0.80697	0.77872	0.72521	0.65190	43
44	0.86379	0.83281	0.80296	0.77420	0.71981	0.64545	44
45	0.86092	0.82935	0.79896	0.76971	0.71445	0.63905	45
46	0.85806	0.82591	0.79499	0.76525	0.70913	0.63273	46
47	0.85521	0.82248	0.79103	0.76081	0.70385	0.62646	47
48	0.85237	0.81907	0.78710	0.75640	0.69861	0.62026	48
49	0.84954	0.81567	0.78318	0.75201	0.69341	0.61412	49
50	0.84672	0.81229	0.77929	0.74765	0.68825	0.60804	50
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE IV

PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
51	0.84390	0.80892	0.77541	0.74331	0.68313	0.60202	51
52	0.84110	0.80556	0.77155	0.73900	0.67804	0.59606	52
53	0.83831	0.80222	0.76771	0.73472	0.67300	0.59016	53
54	0.83552	0.79889	0.76389	0.73046	0.66799	0.58431	54
55	0.83274	0.79557	0.76009	0.72622	0.66301	0.57853	55
56	0.82998	0.79227	0.75631	0.72201	0.65808	0.57280	56
57	0.82722	0.78899	0.75255	0.71782	0.65318	0.56713	57
58	0.82447	0.78571	0.74880	0.71366	0.64832	0.56151	58
59	0.82173	0.78245	0.74508	0.70952	0.64349	0.55595	59
60	0.81900	0.77921	0.74137	0.70541	0.63870	0.55045	60
61	0.81628	0.77597	0.73768	0.70131	0.63395	0.54500	61
62	0.81357	0.77275	0.73401	0.69725	0.62923	0.53960	62
63	0.81087	0.76955	0.73036	0.69320	0.62454	0.53426	63
64	0.80817	0.76635	0.72673	0.68918	0.61989	0.52897	64
65	0.80549	0.76317	0.72311	0.68519	0.61528	0.52373	65
66	0.80281	0.76001	0.71952	0.68121	0.61070	0.51855	66
67	0.80015	0.75685	0.71594	0.67726	0.60615	0.51341	67
68	0.79749	0.75371	0.71237	0.67333	0.60164	0.50833	68
69	0.79484	0.75058	0.70883	0.66943	0.59716	0.50330	69
70	0.79220	0.74747	0.70530	0.66555	0.59272	0.49831	70
71	0.78957	0.74437	0.70179	0.66169	0.58830	0.49338	71
72	0.78694	0.74128	0.69830	0.65785	0.58392	0.48850	72
73	0.78433	0.73820	0.69483	0.65403	0.57958	0.48366	73
74	0.78172	0.73514	0.69137	0.65024	0.57526	0.47887	74
75	0.77912	0.73209	0.68793	0.64647	0.57098	0.47413	75
76	0.77654	0.72905	0.68451	0.64272	0.56673	0.46944	76
77	0.77396	0.72603	0.68110	0.63899	0.56251	0.46479	77
78	0.77139	0.72302	0.67772	0.63529	0.55832	0.46019	78
79	0.76882	0.72002	0.67434	0.63160	0.55417	0.45563	79
80	0.76627	0.71703	0.67099	0.62794	0.55004	0.45112	80
81	0.76372	0.71405	0.66765	0.62430	0.54595	0.44665	81
82	0.76119	0.71109	0.66433	0.62068	0.54188	0.44223	82
83	0.75866	0.70814	0.66102	0.61708	0.53785	0.43785	83
84	0.75614	0.70520	0.65773	0.61350	0.53385	0.43352	84
85	0.75362	0.70227	0.65446	0.60994	0.52987	0.42922	85
86	0.75112	0.69936	0.65121	0.60640	0.52593	0.42497	86
87	0.74862	0.69646	0.64797	0.60289	0.52201	0.42077	87
88	0.74614	0.69357	0.64474	0.59939	0.51813	0.41660	88
89	0.74366	0.69069	0.64154	0.59591	0.51427	0.41248	89
90	0.74119	0.68782	0.63834	0.59246	0.51044	0.40839	90
96	0.72654	0.67088	0.61952	0.57214	0.48806	0.38472	96
100	0.71693	0.65981	0.60729	0.55898	0.47369	0.36971	100
108	0.69809	0.63822	0.58353	0.53357	0.44620	0.34142	108
120	0.67077	0.60716	0.54963	0.49760	0.40791	0.30299	120
132	0.64451	0.57761	0.51770	0.46405	0.37295	0.26889	132
144	0.61928	0.54950	0.48763	0.43277	0.34097	0.23863	144
156	0.59503	0.52275	0.45930	0.40359	0.31172	0.21177	156
168	0.57174	0.49731	0.43262	0.37638	0.28499	0.18794	168
180	0.54936	0.47310	0.40748	0.35101	0.26055	0.16678	180
192	0.52785	0.45008	0.38381	0.32734	0.23820	0.14801	192
200	0.51399	0.43535	0.36880	0.31216	0.22438	0.13669	200
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE IV

PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	1¼%	1½%	1¾%	2%	2½%	3%	n
1	0.98765	0.98522	0.98280	0.98039	0.97561	0.97087	1
2	0.97546	0.97066	0.96590	0.96117	0.95181	0.94260	2
3	0.96342	0.95632	0.94929	0.94232	0.92860	0.91514	3
4	0.95152	0.94218	0.93296	0.92385	0.90595	0.88849	4
5	0.93978	0.92826	0.91691	0.90573	0.88385	0.86261	5
6	0.92817	0.91454	0.90114	0.88797	0.86230	0.83748	6
7	0.91672	0.90103	0.88564	0.87056	0.84127	0.81309	7
8	0.90540	0.88771	0.87041	0.85349	0.82075	0.78941	8
9	0.89422	0.87459	0.85544	0.83676	0.80073	0.76642	9
10	0.88318	0.86167	0.84073	0.82035	0.78120	0.74409	10
11	0.87228	0.84893	0.82627	0.80426	0.76214	0.72242	11
12	0.86151	0.83639	0.81206	0.78849	0.74356	0.70138	12
13	0.85087	0.82403	0.79809	0.77303	0.72542	0.68095	13
14	0.84037	0.81185	0.78436	0.75788	0.70773	0.66112	14
15	0.82999	0.79985	0.77087	0.74301	0.69047	0.64186	15
16	0.81975	0.78803	0.75762	0.72845	0.67362	0.62317	16
17	0.80963	0.77639	0.74459	0.71416	0.65720	0.60502	17
18	0.79963	0.76491	0.73178	0.70016	0.64117	0.58739	18
19	0.78976	0.75361	0.71919	0.68643	0.62553	0.57029	19
20	0.78001	0.74247	0.70682	0.67297	0.61027	0.55368	20
21	0.77038	0.73150	0.69467	0.65978	0.59539	0.53755	21
22	0.76087	0.72069	0.68272	0.64684	0.58086	0.52189	22
23	0.75147	0.71004	0.67098	0.63416	0.56670	0.50669	23
24	0.74220	0.69954	0.65944	0.62172	0.55288	0.49193	24
25	0.73303	0.68921	0.64810	0.60953	0.53939	0.47761	25
26	0.72398	0.67902	0.63695	0.59758	0.52623	0.46369	26
27	0.71505	0.66899	0.62599	0.58586	0.51340	0.45019	27
28	0.70622	0.65910	0.61523	0.57437	0.50088	0.43708	28
29	0.69750	0.64936	0.60465	0.56311	0.48866	0.42435	29
30	0.68889	0.63976	0 59425	0.55207	0.47674	0.41199	30
31	0.68038	0.63031	0.58403	0.54125	0.46511	0.39999	31
32	0.67198	0.62099	0.57398	0.53063	0.45377	0.38834	32
33	0.66369	0.61182	0.56411	0.52023	0.44270	0.37703	33
34	0.65549	0.60277	0.55441	0.51003	0.43191	0.36604	34
35	0.64740	0.59387	0.54487	0.50003	0.42137	0.35538	35
36	0.63941	0.58509	0.53550	0.49022	0.41109	0.34503	36
37	0.63152	0.57644	0.52629	0.48061	0.40107	0.33498	37
38	0.62372	0.56792	0.51724	0.47119	0.39128	0.32523	38
39	0.61602	0.55953	0.50834	0.46195	0.38174	0.31575	39
40	0.60841	0.55126	0.49960	0.45289	0.37243	0.30656	40
44	0.57892	0.51939	0.46611	0.41840	0.33740	0.27237	44
48	0.55086	0.48936	0.43486	0.38654	0.30567	0.24200	48
50	0.53734	0.47500	0.42003	0.37153	0.29094	0.22811	50
52	0.52415	0.46107	0.40570	0.35710	0.27692	0.21501	52
56	0.49874	0.43441	0.37851	0.32991	0.25088	0.19104	56
60	0.47457	0.40930	0.35313	0.30478	0.22728	0.16973	60
64	0.45156	0.38563	0.32946	0.28157	0.20591	0.15081	64
68	0.42967	0.36334	0.30737	0.26013	0.18654	0.13399	68
72	0.40884	0.34233	0.28676	0.24032	0.16900	0.11905	72
76	0.38903	0.32254	0.26754	0.22202	0.15310	0.10577	76
80	0.37017	0.30389	0.24960	0.20511	0.13870	0.09398	80
n	1¼%	1½%	1¾%	2%	2½%	3%	n

TABLE IV

PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n
1	0.96618	0.96154	0.95694	0.95238	0.94340	0.93458	1
2	0.93351	0.92455	0.91573	0.90703	0.89000	0.87344	2
3	0.90194	0.88900	0.87630	0.86384	0.83962	0.81630	3
4	0.87144	0.85480	0.83856	0.82270	0.79209	0.76290	4
5	0.84197	0.82193	0.80245	0.78353	0.74726	0.71299	5
6	0.81350	0.79031	0.76790	0.74622	0.70496	0.66634	6
7	0.78599	0.75992	0.73483	0.71068	0.66506	0.62275	7
8	0.75941	0.73069	0.70319	0.67684	0.62741	0.58201	8
9	0.73373	0.70259	0.67290	0.64461	0.59190	0.54393	9
10	0.70892	0.67556	0.64393	0.61391	0.55839	0.50835	10
11	0.68495	0.64958	0.61620	0.58468	0.52679	0.47509	11
12	0.66178	0.62460	0.58966	0.55684	0.49697	0.44401	12
13	0.63940	0.60057	0.56427	0.53032	0.46884	0.41496	13
14	0.61778	0.57748	0.53997	0.50507	0.44230	0.38782	14
15	0.59689	0.55526	0.51672	0.48102	0.41727	0.36245	15
16	0.57671	0.53391	0.49447	0.45811	0.39365	0.33873	16
17	0.55720	0.51337	0.47318	0.43630	0.37136	0.31657	17
18	0.53836	0.49363	0.45280	0.41552	0.35034	0.29586	18
19	0.52016	0.47464	0.43330	0.39573	0.33051	0.27651	19
20	0.50257	0.45639	0.41464	0.37689	0.31180	0.25842	20
21	0.48557	0.43883	0.39679	0.35894	0.29416	0.24151	21
22	0.46915	0.42196	0.37970	0.34185	0.27751	0.22571	22
23	0.45329	0.40573	0.36335	0.32557	0.26180	0.21095	23
24	0.43796	0.39012	0.34770	0.31007	0.24698	0.19715	24
25	0.42315	0.37512	0.33273	0.29530	0.23300	0.18425	25
26	0.40884	0.36069	0.31840	0.28124	0.21981	0.17220	26
27	0.39501	0.34682	0.30469	0.26785	0.20737	0.16093	27
28	0.38165	0.33348	0.29157	0.25509	0.19563	0.15040	28
29	0.36875	0.32065	0.27902	0.24295	0.18456	0.14056	29
30	0.35628	0.30832	0.26700	0.23138	0.17411	0.13137	30
31	0.34423	0.29646	0.25550	0.22036	0.16425	0.12277	31
32	0.33259	0.28506	0.24450	0.20987	0.15496	0.11474	32
33	0.32134	0.27409	0.23397	0.19987	0.14619	0.10723	33
34	0.31048	0.26355	0.22390	0.19035	0.13791	0.10022	34
35	0.29998	0.25342	0.21425	0.18129	0.13011	0.09366	35
36	0.28983	0.24367	0.20503	0.17266	0.12274	0.08751	36
37	0.28003	0.23430	0.19620	0.16444	0.11579	0.08181	37
38	0.27056	0.22529	0.18775	0.15661	0.10924	0.07646	38
39	0.26141	0.21662	0.17967	0.14915	0.10306	0.07146	39
40	0.25257	0.20829	0.17193	0.14205	0.09722	0.06678	40
41	0.24403	0.20028	0.16453	0.13528	0.09172	0.06241	41
42	0.23578	0.19257	0.15744	0.12884	0.08653	0.05833	42
43	0.22781	0.18517	0.15066	0.12270	0.08163	0.05451	43
44	0.22010	0.17805	0.14417	0.11686	0.07701	0.05095	44
45	0.21266	0.17120	0.13796	0.11130	0.07265	0.04761	45
46	0.20547	0.16461	0.13202	0.10600	0.06854	0.04450	46
47	0.19852	0.15828	0.12634	0.10095	0.06466	0.04159	47
48	0.19181	0.15219	0.12090	0.09614	0.06100	0.03887	48
49	0.18532	0.14631	0.11569	0.09156	0.05755	0.03632	49
50	0.17905	0.14071	0.11071	0.08720	0.05429	0.03395	50
n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n

TABLE V

ACCUMULATED VALUE OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
2	2.00333	2.00417	2.00500	2.00583	2.00750	2.01000	2
3	3.01001	3.01252	3.01502	3.01753	3.02256	3.03010	3
4	4.02004	4.02507	4.03010	4.03514	4.04523	4.06040	4
5	5.03344	5.04184	5.05025	5.05867	5.07556	5.10101	5
6	6.05022	6.06285	6.07550	6.08818	6.11363	6.15202	6
7	7.07039	7.08811	7.10588	7.12370	7.15948	7.21354	7
8	8.09396	8.11764	8.14141	8.16525	8.21318	8.28567	8
9	9.12094	9.15147	9.18212	9.21288	9.27478	9.36853	9
10	10.15134	10.18960	10.22803	10.26663	10.34434	10.46221	10
11	11.1852	11.2321	11.2792	11.3265	11.4219	11.5668	11
12	12.2225	12.2789	12.3356	12.3926	12.5076	12.6825	12
13	13.2632	13.3300	13.3972	13.4649	13.6014	13.8093	13
14	14.3074	14.3856	14.4642	14.4534	14.7034	14.9474	14
15	15.3551	15.4455	15.5365	15.6283	15.8137	16.0969	15
16	16.4063	16.5099	16.6142	16.7194	16.9323	17.2579	16
17	17.4610	17.5786	17.6973	17.8170	18.0593	18.4304	17
18	18.5192	18.6519	18.7858	18.9209	19.1947	19.6147	18
19	19.5809	19.7296	19.8797	20.0313	20.3387	20.8109	19
20	20.6462	20.8118	20.9791	21.1481	21.4912	22.0190	20
21	21.7150	21.8985	22.0840	22.2715	22.6524	23.2392	21
22	22.7874	22.9898	23.1944	23.4014	23.8223	24.4716	22
23	23.8633	24.0856	24.3104	24.5379	25.0010	25.7163	23
24	24.9429	25.1859	25.4320	25.6810	26.1885	26.9735	24
25	26.0260	26.2909	26.5591	26.8308	27.3849	28.2432	25
26	27.1128	27.4004	27.6919	27.9874	28.5903	29.5256	26
27	28.2032	28.5146	28.8304	29.1506	29.8047	30.8209	27
28	29.2972	29.6334	29.9745	30.3207	31.0282	32.1291	28
29	30.3948	30.7569	31.1244	31.4975	32.2609	33.4504	29
30	31.4961	31.8850	32.2800	32.6813	33.5029	34.7849	30
31	32.6011	33.0179	33.4414	33.8719	34.7542	36.1327	31
32	33.7098	34.1554	34.6086	35.0695	36.0148	37.4941	32
33	34.8222	35.2978	35.7817	36.2741	37.2849	38.8690	33
34	35.9382	36.4448	36.9606	37.4857	38.5646	40.2577	34
35	37.0580	37.5967	38.1454	38.7043	39.8538	41.6603	35
36	38.1816	38.7533	39.3361	39.9301	41.1527	43.0769	36
37	39.3088	39.9148	40.5328	41.1630	42.4614	44.5076	37
38	40.4399	41.0811	41.7354	42.4031	43.7798	45.9527	38
39	41.5747	42.2523	42.9441	43.6505	45.1082	47.4123	39
40	42.7132	43.4283	44.1588	44.9051	46.4465	48.8864	40
41	43.8556	44.6093	45.3796	46.1671	47.7948	50.3752	41
42	45.0018	45.7952	46.6065	47.4364	49.1533	51.8790	42
43	46.1518	46.9860	47.8396	48.7131	50.5219	53.3978	43
44	47.3057	48.1818	49.0788	49.9972	51.9009	54.9318	44
45	48.4633	49.3825	50.3242	51.2889	53.2901	56.4811	45
46	49.6249	50.5883	51.5758	52.5881	54.6898	58.0459	46
47	50.7903	51.7991	52.8337	53.8948	56.1000	59.6263	47
48	51.9596	53.0149	54.0978	55.2092	57.5207	61.2226	48
49	53.1328	54.2358	55.3683	56.5310	58.9521	62.8348	49
50	54.3099	55.4618	56.6452	57.8611	60.3943	64.4632	50
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE V

ACCUMULATED VALUE OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
51	55.4909	56.6929	57.9284	59.1986	61.8472	66.1078	51
52	56.6759	57.9291	59.2180	60.5439	63.3111	67.7689	52
53	57.8648	59.1704	60.5141	61.8971	64.7859	69.4466	53
54	59.0577	60.4170	61.8167	63.2581	66.2718	71.1410	54
55	60.2546	61.6687	63.1258	64.6271	67.7688	72.8525	55
56	61.4554	62.9257	64.4414	66.0041	69.2771	74.5810	56
57	62.6603	64.1879	65.7636	67.3892	70.7967	76.3268	57
58	63.8691	65.4553	67.0924	68.7823	72.3277	78.0901	58
59	65.0820	66.7280	68.4279	70.1835	73.8701	79.8710	59
60	66.2990	68.0061	69.7700	71.5929	75.4241	81.6697	60
61	67.5200	69.2894	71.1189	73.0105	76.9898	83.4864	61
62	68.7450	70.5781	72.4745	74.4364	78.5672	85.3212	62
63	69.9742	71.8722	73.8368	75.8706	80.1565	87.1744	63
64	71.2074	73.1717	75.2060	77.3132	81.7577	89.0462	64
65	72.4448	74.4766	76.5821	78.7642	83.3709	90.9366	65
66	73.6863	75.7869	77.9650	80.2237	84.9961	92.8460	66
67	74.9319	77.1027	79.3548	81.6916	86.6336	94.7745	67
68	76.1817	78.4240	80.7516	83.1682	88.2834	96.7222	68
69	77.4356	79.7507	82.1553	84.6533	89.9455	98.6894	69
70	78.6937	81.0830	83.5661	86.1471	91.6201	100.6763	70
71	79.9560	82.4208	84.9839	87.6497	93.3072	102.6831	71
72	81.2226	83.7643	86.4089	89.1609	95.0070	104.7099	72
73	82.4933	85.1133	87.8409	90.6810	96.7196	106.7570	73
74	83.7683	86.4679	89.2801	92.2100	98.4450	108.8246	74
75	85.0475	87.8282	90.7265	93.7479	100.1833	110.9128	75
76	86.3310	89.1941	92.1801	95.2948	101.9347	113.0220	76
77	87.6188	90.5658	93.6410	96.8507	103.6992	115.1522	77
78	88.9108	91.9431	95.1092	98.4156	105.4769	117.3037	78
79	90.2072	93.3262	96.5848	99.9897	107.2680	119.4768	79
80	91.5079	94.7151	98.0677	101.5730	109.0725	121.6715	80
81	92.8129	96.1098	99.5581	103.1655	110.8906	123.8882	81
82	94.1223	97.5102	101.0558	104.7673	112.7223	126.1271	82
83	95.4360	98.9165	102.5611	106.3784	114.5677	128.3884	83
84	96.7542	100.3287	104.0739	107.9990	116.4269	130.6723	84
85	98.0767	101.7467	105.5943	109.6290	118.3001	132.9790	85
86	99.4036	103.1706	107.1223	111.2685	120.1874	135.3088	86
87	100.7349	104.6005	108.6579	112.9175	122.0888	137.6619	87
88	102.0707	106.0363	110.2012	114.5762	124.0045	140.0385	88
89	103.4110	107.4782	111.7522	116.2446	125.9345	142.4389	89
90	104.7557	108.9260	113.3109	117.9227	127.8790	144.8633	90
96	112.919	117.741	122.829	128.199	139.856	159.927	96
100	118.452	123.740	129.334	135.252	148.145	170.481	100
108	129.741	136.043	142.740	149.859	165.183	192.893	108
120	147.250	155.282	163.879	173.085	193.514	230.039	120
132	165.471	175.506	186.323	197.990	224.175	271.896	132
144	184.435	196.764	210.150	224.695	257.712	319.062	144
156	204.172	219.109	235.447	253.331	294.394	372.209	156
168	224.713	242.598	262.305	284.037	334.518	432.097	168
180	246.090	267.289	290.819	316.962	378.406	499.580	180
192	268.339	293.243	321.091	352.268	426.410	575.622	192
200	283.673	311.279	342.303	377.212	460.890	631.602	200
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE V

ACCUMULATED VALUE OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
2	2.01250	2.01500	2.01750	2.02000	2.02500	2.03000	2
3	3.03766	3.04522	3.05281	3.06040	3.07562	3.09090	3
4	4.07563	4.09090	4.10623	4.12161	4.15252	4.18363	4
5	5.12657	5.15227	5.17809	5.20404	5.25633	5.30914	5
6	6.19065	6.22955	6.26871	6.30812	6.38774	6.46841	6
7	7.26804	7.32299	7.37841	7.43428	7.54743	7.66246	7
8	8.35889	8.43284	8.50753	8.58297	8.73612	8.89234	8
9	9.46337	9.55933	9.65641	9.75463	9.95452	10.15911	9
10	10.58167	10.70272	10.82540	10.94972	11.20338	11.46388	10
11	11.7139	11.8633	12.0148	12.1687	12.4835	12.8078	11
12	12.8604	13.0412	13.2251	13.4121	13.7956	14.1920	12
13	14.0211	14.2368	14.4565	14.6803	15.1404	15.6178	13
14	15.1964	15.4504	15.7095	15.9739	16.5190	17.0863	14
15	16.3863	16.6821	16.9844	17.2934	17.9319	18.5989	15
16	17.5912	17.9324	18.2817	18.6393	19.3802	20.1569	16
17	18.8111	19.2014	19.6016	20.0121	20.8647	21.7616	17
18	20.0462	20.4894	20.9446	21.4123	22.3863	23.4144	18
19	21.2968	21.7967	22.3112	22.8406	23.9460	25.1169	19
20	22.5630	23.1237	23.7016	24.2974	25.5447	26.8704	20
21	23.8450	24.4705	25.1164	25.7833	27.1833	28.6765	21
22	25.1431	25.8376	26.5559	27.2990	28.8629	30.5368	22
23	26.4574	27.2251	28.0207	28.8450	30.5844	32.4529	23
24	27.7881	28.6335	29.5110	30.4219	32.3490	34.4265	24
25	29.1354	20.0630	31.0275	32.0303	34.1578	36.4593	25
26	30.4996	31.5140	32.5704	33.6709	36.0117	38.5530	26
27	31.8809	32.9867	34.1404	35.3443	37.9120	40.7096	27
28	33.2794	34.4815	35.7379	37.0512	39.8598	42.9309	28
29	34.6954	35.9987	37.3633	38.7922	41.8563	45.2189	29
30	36.1291	37.5387	39.0172	40.5681	43.9027	47.5754	30
31	37.5807	39.1018	40.7000	42.3794	46.0003	50.0027	31
32	39.0504	40.6883	42.4122	44.2270	48.1503	52.5028	32
33	40.5386	42.2986	44.1541	46.1116	50.3540	55.0778	33
34	42.0453	43.9331	45.9271	48.0338	52.6129	57.7302	34
35	43.5709	45.5921	47.7308	49.9945	54.9282	60.4621	35
36	45.1155	47.2760	49.5661	51.9944	57.3014	63.2759	36
37	46.6794	48.9851	51.4335	54.0343	59.7339	66.1742	37
38	48.2629	50.7199	53.3336	56.1149	62.2273	69.1594	38
39	49.8662	52.4807	55.2670	58.2372	64.7830	72.2342	39
40	51.4896	54.2679	57.2341	60.4020	67.4026	75.4013	40
44	58.1883	61.6889	65.4532	69.5027	78.5523	89.0484	44
48	65.2284	69.5652	74.2628	79.3535	90.8596	104.4084	48
50	68.8818	73.6828	78.9022	84.5794	97.4843	112.7969	50
52	72.6271	77.9249	83.7055	90.0164	104.4445	121.6962	52
56	80.4027	86.7975	93.8267	101.5583	119.4397	141.1538	56
60	88.5745	96.2147	104.6752	114.0515	135.9916	163.0534	60
64	97.1626	106.2096	116.3033	127.5747	154.2618	187.7017	64
68	106.1882	116.8179	128.7670	142.2125	174.4287	215.4436	68
72	115.6736	128.0772	142.1263	158.0570	196.6891	246.6672	72
76	125.6423	140.0274	156.4456	175.2076	221.2605	281.8098	76
80	136.1188	152.7109	171.7938	193.7720	248.3827	321.3630	80
n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n

TABLE V

ACCUMULATED VALUE OF ANNUITY OF 1 PER PERIOD

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1
2	2.03500	2.04000	2.04500	2.05000	2.06000	2.07000	2
3	3.10622	3.12160	3.13702	3.15250	3.18360	3.21490	3
4	4.21494	4.24646	4.27819	4.31012	4.37462	4.43994	4
5	5.36247	5.41632	5.47071	5.52563	5.63709	5.75074	5
6	6.55015	6.63298	6.71689	6.80191	6.97532	7.15329	6
7	7.77941	7.89829	8.01915	8.14201	8.39384	8.65402	7
8	9.05169	9.21423	9.38001	9.54911	9.89747	10.25980	8
9	10.36850	10.58280	10.80211	11.02656	11.49132	11.97799	9
10	11.73139	12.00611	12.28821	12.57789	13.18079	13.81645	10
11	13.1420	13.4864	13.8412	14.2068	14.9716	15.7836	11
12	14.6020	15.0258	15.4640	15.9171	16.8699	17.8885	12
13	16.1130	16.6268	17.1599	17.7130	18.8821	20.1406	13
14	17.6770	18.2919	18.9321	19.5986	21.0151	22.5505	14
15	19.2957	20.0236	20.7841	21.5786	23.2760	25.1290	15
16	20.9710	21.8245	22.7193	23.6575	25.6725	27.8881	16
17	22.7050	23.6975	24.7417	25.8404	28.2129	30.8402	17
18	24.4997	25.6454	26.8551	28.1324	30.9057	33.9990	18
19	26.3572	27.6712	29.0636	30.5390	33.7600	37.3790	19
20	28.2797	29.7781	31.3714	33.0660	36.7856	40.9955	20
21	30.2695	31.9692	33.7831	35.7193	39.9927	44.8652	21
22	32.3289	34.2480	36.3034	38.5052	43.3923	49.0057	22
23	34.4604	36.6179	38.9370	41.4305	46.9958	53.4361	23
24	36.6665	39.0826	41.6892	44.5020	50.8156	58.1767	24
25	38.9499	41.6459	44.5652	47.7271	54.8645	63.2490	25
26	41.3131	44.3117	47.5706	51.1135	59.1564	68.6765	26
27	43.7591	47.0842	50.7113	54.6691	63.7058	74.4838	27
28	46.2906	49.9676	53.9933	58.4026	68.5281	80.6977	28
29	48.9108	52.9663	57.4230	62.3227	73.6398	87.3465	29
30	51.6227	56.0849	61.0071	66.4388	79.0582	94.4608	30
31	54.4295	59.3283	64.7524	70.7608	84.8017	102.073	31
32	57.3345	62.7015	68.6662	75.2988	90.8898	110.218	32
33	60.3412	66.2095	72.7562	80.0638	97.3432	118.933	33
34	63.4532	69.8579	77.0303	85.0670	104.1838	128.259	34
35	66.6740	73.6522	81.4966	90.3203	111.4348	138.237	35
36	70.0076	77.5983	86.1640	95.8363	119.121	148.913	36
37	73.4579	81.7022	91.0413	101.6281	127.268	160.337	37
38	77.0289	85.9703	96.1382	107.7095	135.904	172.561	38
39	80.7249	90.4091	101.4644	114.0950	145.058	185.640	39
40	84.5503	95.0255	107.0303	120.7998	154.762	199.635	40
41	88.5095	99.8265	112.847	127.840	165.048	214.610	41
42	92.6074	104.8196	118.925	135.232	175.951	230.632	42
43	96.8486	110.0124	125.276	142.993	187.508	247.776	43
44	101.2383	115.4129	131.914	151.143	199.758	266.121	44
45	105.7817	121.0294	138.850	159.700	212.744	285.749	45
46	110.484	126.871	146.098	168.685	226.508	306.752	46
47	115.351	132.945	153.673	178.119	241.099	329.224	47
48	120.388	139.263	161.588	188.025	256.565	353.270	48
49	125.602	145.834	169.859	198.427	272.958	378.999	49
50	130.998	152.667	178.503	209.348	290.336	406.529	50
n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n

TABLE VI

PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
1	0.99668	0.99585	0.99502	0.99420	0.99256	0.99010	1
2	1.99004	1.98757	1.98510	1.98264	1.97772	1.97040	2
3	2.98011	2.97517	2.97025	2.96534	2.95556	2.94099	3
4	3.96689	3.95868	3.95050	3.94234	3.92611	3.90197	4
5	4.95039	4.93810	4.92587	4.91368	4.88944	4.85343	5
6	5.93062	5.91346	5.89638	5.87938	5.84560	5.79548	6
7	6.90759	6.88478	6.86207	6.83948	6.79464	6.72819	7
8	7.88132	7.85206	7.82296	7.79402	7.73661	7.65168	8
9	8.85182	8.81533	8.77906	8.74302	8.67158	8.56602	9
10	9.81908	9.77460	9.73041	9.68561	9.59958	9.47130	10
11	10.7831	10.7299	10.6770	10.6245	10.5207	10.3676	11
12	11.7440	11.6812	11.6189	11.5571	11.4349	11.2551	12
13	12.7017	12.6286	12.5562	12.4843	12.3423	12.1337	13
14	13.6561	13.5721	13.4887	13.4061	13.2430	13.0037	14
15	14.6074	14.5116	14.4166	14.3225	14.1370	13.8651	15
16	15.5556	15.4472	15.3399	15.2337	15.0243	14.7179	16
17	16.5006	16.3790	16.2586	16.1395	15.9050	15.5623	17
18	17.4424	17.3069	17.1728	17.0401	16.7792	16.3983	18
19	18.3812	18.2309	18.0824	17.9355	17.6468	17.2260	19
20	19.3168	19.1511	18.9874	18.8257	18.5080	18.0456	20
21	20.2493	20.0675	19.8880	19.7107	19.3628	18.8570	21
22	21.1787	20.9801	20.7841	20.5906	20.2112	19.6604	22
23	22.1050	21.8889	21.6757	21.4654	21.0533	20.4558	23
24	23.0283	22.7939	22.5629	22.3351	21.8891	21.2434	24
25	23.9484	23.6952	23.4456	23.1998	22.7188	22.0232	25
26	24.8655	24.5927	24.3240	24.0594	23.5422	22.7952	26
27	25.7796	25.4865	25.1980	24.9141	24.3595	23.5596	27
28	26.6906	26.3766	26.0677	25.7638	25.1707	24.3164	28
29	27.5986	27.2630	26.9330	26.6086	25.9759	25.0658	29
30	28.5036	28.1457	27.7941	27.4485	26.7751	25.8077	30
31	29.4056	29.0248	28.6508	28.2835	27.5683	26.5423	31
32	30.3046	29.9002	29.5033	29.1137	28.3557	27.2696	32
33	31.2006	30.7720	30.3515	29.9390	29.1371	27.9897	33
34	32.0936	31.6402	31.1955	30.7596	29.9128	28.7027	34
35	32.9837	32.5047	32.0354	31.5754	30.6827	29.4086	35
36	33.8708	33.3657	32.8710	32.3865	31.4468	30.1075	36
37	34.7549	34.2231	33.7025	33.1928	32.2053	30.7995	37
38	35.6361	35.0770	34.5299	33.9945	32.9581	31.4847	38
39	36.5144	35.9273	35.3531	34.7916	33.7053	32.1630	39
40	37.3898	36.7740	36.1722	35.5840	34.4469	32.8347	40
41	38.2622	37.6173	36.9873	36.3718	35.1831	33.4997	41
42	39.1318	38.4571	37.7983	37.1551	35.9137	34.1581	42
43	39.9985	39.2933	38.6053	37.9338	36.6389	34.8100	43
44	40.8623	40.1261	39.4082	38.7080	37.3587	35.4555	44
45	41.7232	40.9555	40.2072	39.4777	38.0732	36.0945	45
46	42.5813	41.7814	41.0022	40.2430	38.7823	36.7272	46
47	43.4365	42.6039	41.7932	41.0038	39.4862	37.3537	47
48	44.2888	43.4230	42.5803	41.7602	40.1848	37.9740	48
49	45.1384	44.2386	43.3635	42.5122	40.8782	38.5881	49
50	45.9851	45.0509	44.1428	43.2599	41.5664	39.1961	50
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE VI

PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n
51	46.8290	45.8598	44.9182	44.0032	42.2496	39.7981	51
52	47.6701	46.6654	45.6897	44.7422	42.9276	40.3942	52
53	48.5084	47.4676	46.4575	45.4769	43.6006	40.9844	53
54	49.3439	48.2665	47.2214	46.2074	44.2686	41.5687	54
55	50.1767	49.0621	47.9814	46.9336	44.9316	42.1472	55
56	51.0066	49.8544	48.7378	47.6556	45.5897	42.7200	56
57	51.8339	50.6433	49.4903	48.3734	46.2429	43.2871	57
58	52.6583	51.4290	50.2391	49.0871	46.8912	43.8486	58
59	53.4801	52.2115	50.9842	49.7966	47.5347	44.4046	59
60	54.2991	52.9907	51.7256	50.5020	48.1734	44.9550	60
61	55.1154	53.7667	52.4632	51.2033	48.8073	45.5000	61
62	55.9289	54.5394	53.1973	51.9006	49.4365	46.0396	62
63	56.7398	55.3090	53.9276	52.5938	50.0611	46.5739	63
64	57.5480	56.0753	54.6543	53.2829	50.6810	47.1029	64
65	58.3535	56.8385	55.3775	53.9681	51.2963	47.6266	65
66	59.1563	57.5985	56.0970	54.6493	51.9070	48.1452	66
67	59.9564	58.3554	56.8129	55.3266	52.5131	48.6586	67
68	60.7539	59.1091	57.5253	55.9999	53.1147	49.1669	68
69	61.5487	59.8597	58.2341	56.6694	53.7119	49.6702	69
70	62.3409	60.6071	58.9394	57.3349	54.3046	50.1685	70
71	63.1305	61.3515	59.6412	57.9966	54.8929	50.6619	71
72	63.9174	62.0928	60.3395	58.6544	55.4768	51.1504	72
73	64.7018	62.8310	61.0343	59.3085	56.0564	51.6341	73
74	65.4835	63.5661	61.7257	59.9587	56.6317	52.1129	74
75	66.2626	64.2982	62.4136	60.6052	57.2027	52.5871	75
76	67.0391	65.0273	63.0982	61.2479	57.7694	53.0565	76
77	67.8131	65.7533	63.7793	61.8869	58.3319	53.5213	77
78	68.5845	66.4763	64.4570	62.5222	58.8902	53.9815	78
79	69.3533	67.1963	65.1313	63.1538	59.4444	54.4371	79
80	70.1196	67.9134	65.8023	63.7817	59.9944	54.8882	80
81	70.8833	68.6274	66.4700	64.4060	60.5404	55.3349	81
82	71.6445	69.3385	67.1343	65.0267	61.0823	55.7771	82
83	72.4031	70.0466	67.7953	65.6438	61.6201	56.2149	83
84	73.1593	70.7518	68.4530	66.2573	62.1540	56.6485	84
85	73.9129	71.4541	69.1075	66.8672	62.6838	57.0777	85
86	74.6640	72.1535	69.7587	67.4736	63.2098	57.5026	86
87	75.4126	72.8499	70.4067	68.0765	63.7318	57.9234	87
88	76.1588	73.5435	71.0514	68.6759	64.2499	58.3400	88
89	76.9024	74.2342	71.6930	69.2718	64.7642	58.7525	89
90	77.6436	74.9220	72.3313	69.8643	65.2746	59.1609	90
96	82.0393	78.9894	76.0952	73.3476	68.2584	61.5277	96
100	84.9214	81.6452	78.5426	75.6031	70.1746	63.0289	100
108	90.5718	86.8261	83.2934	79.9598	73.8394	65.8578	108
120	98.7702	94.2814	90.0735	86.1264	78.9417	69.7005	120
132	106.648	101.374	96.4596	91.8771	83.6064	73.1108	132
144	114.217	108.121	102.475	97.2402	87.8711	76.1372	144
156	121.490	114.540	108.140	102.242	91.7700	78.8229	156
168	128.478	120.646	113.477	106.906	95.3346	81.2064	168
180	135.192	126.455	118.504	111.256	98.5934	83.3217	180
192	141.644	131.982	123.238	115.313	101.573	85.1988	192
200	145.804	135.516	126.211	117.861	103.416	86.3314	200
n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%	n

TABLE VI

PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.98765	0.98522	0.98280	0.98039	0.97561	0.97087	1
2	1.96312	1.95588	1.94870	1.94156	1.92742	1.91347	2
3	2.92653	2.91220	2.89798	2.88388	2.85602	2.82861	3
4	3.87806	3.85438	3.83094	3.80773	3.76197	3.71710	4
5	4.81784	4.78264	4.74786	4.71346	4.64583	4.57971	5
6	5.74601	5.69719	5.64900	5.60143	5.50813	5.41719	6
7	6.66273	6.59821	6.53464	6.47199	6.34939	6.23028	7
8	7.56812	7.48593	7.40505	7.32548	7.17014	7.01969	8
9	8.46234	8.36052	8.26049	8.16224	7.97087	7.78611	9
10	9.34553	9.22218	9.10122	8.98259	8.75206	8.53020	10
11	10.2178	10.0711	9.92749	9.78685	9.51421	9.25262	11
12	11.0793	10.9075	10.73955	10.57534	10.25776	9.95400	12
13	11.9302	11.7315	11.53764	11.34837	10.98318	10.63496	13
14	12.7706	12.5434	12.32201	12.10625	11.69091	11.29607	14
15	13.6005	13.3432	13.09288	12.84926	12.38138	11.93794	15
16	14.4203	14.1313	13.8505	13.5777	13.0550	12.5611	16
17	15.2299	14.9076	14.5951	14.2919	13.7122	13.1661	17
18	16.0295	15.6726	15.3269	14.9920	14.3534	13.7535	18
19	16.8193	16.4262	16.0461	15.6785	14.9789	14.3238	19
20	17.5993	17.1686	16.7529	16.3514	15.5892	14.8775	20
21	18.3697	17.9001	17.4475	17.0112	16.1845	15.4150	21
22	19.1306	18.6208	18.1303	17.6580	16.7654	15.9369	22
23	19.8820	19.3309	18.8012	18.2922	17.3321	16.4436	23
24	20.6242	20.0304	19.4607	18.9139	17.8850	16.9355	24
25	21.3573	20.7196	20.1088	19.5235	18.4244	17.4131	25
26	22.0813	21.3986	20.7457	20.1210	18.9506	17.8768	26
27	22.7963	22.0676	21.3717	20.7069	19.4640	18.3270	27
28	23.5025	22.7267	21.9870	21.2813	19.9649	18.7641	28
29	24.2000	23.3761	22.5916	21.8444	20.4535	19.1885	29
30	24.8889	24.0158	23.1858	22.3965	20.9303	19.6004	30
31	25.5693	24.6461	23.7699	22.9377	21.3954	20.0004	31
32	26.2413	25.2671	24.3439	23.4683	21.8492	20.3888	32
33	26.9050	25.8790	24.9080	23.9886	22.2919	20.7658	33
34	27.5605	26.4817	25.4624	24.4986	22.7238	21.1318	34
35	28.2079	27.0756	26.0073	24.9986	23.1452	21.4872	35
36	28.8473	27.6607	26.5428	25.4888	23.5563	21.8323	36
37	29.4788	28.2371	27.0690	25.9695	23.9573	22.1672	37
38	30.1025	28.8051	27.5863	26.4406	24.3486	22.4925	38
39	30.7185	29.3646	28.0946	26.9026	24.7303	22.8082	39
40	31.3269	29.9158	28.5942	27.3555	25.1028	23.1148	40
44	33.6864	32.0406	30.5082	29.0800	26.5038	24.2543	44
48	35.9315	34.0426	32.2938	30.6731	27.7732	25.2667	48
50	37.0129	34.9997	33.1412	31.4236	28.3623	25.7298	50
52	38.0677	35.9287	33.9597	32.1449	28.9231	26.1662	52
56	40.1004	37.7059	35.5140	33.5047	29.9649	26.9655	56
60	42.0346	39.3803	36.9640	34.7609	30.9087	27.6756	60
64	43.8750	40.9579	38.3168	35.9214	31.7637	28.3065	64
68	45.6262	42.4442	39.5789	36.9936	32.5383	28.8670	68
72	47.2925	43.8447	40.7564	37.9841	33.2401	29.3651	72
76	48.8780	45.1641	41.8550	38.8991	33.8758	29.8076	76
80	50.3867	46.4073	42.8799	39.7445	34.4518	30.2008	80
n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n

TABLE VI

PRESENT VALUE OF ANNUITY OF 1 PER PERIOD

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n
1	0.96618	0.96154	0.95694	0.95238	0.94340	0.93458	1
2	1.89969	1.88609	1.87267	1.85941	1.83339	1.80802	2
3	2.80164	2.77509	2.74896	2.72325	2.67301	2.62432	3
4	3.67308	3.62990	3.58753	3.54595	3.46511	3.38721	4
5	4.51505	4.45182	4.38998	4.32948	4.21236	4.10020	5
6	5.32855	5.24214	5.15787	5.07569	4.91732	4.76654	6
7	6.11454	6.00205	5.89270	5.78637	5.58238	5.38929	7
8	6.87396	6.73274	6.59589	6.46321	6.20979	5.97130	8
9	7.60769	7.43533	7.26879	7.10782	6.80169	6.51523	9
10	8.31661	8.11090	7.91272	7.72173	7.36009	7.02358	10
11	9.00155	8.76048	8.52892	8.30641	7.88687	7.49867	11
12	9.66333	9.38507	9.11858	8.86325	8.38384	7.94269	12
13	10.30274	9.98565	9.68285	9.39357	8.85268	8.35765	13
14	10.92052	10.56312	10.22283	9.89864	9.29498	8.74547	14
15	11.51741	11.11839	10.73955	10.37966	9.71225	9.10791	15
16	12.0941	11.6523	11.2340	10.8378	10.1059	9.44665	16
17	12.6513	12.1657	11.7072	11.2741	10.4773	9.76322	17
18	13.1897	12.6593	12.1600	11.6896	10.8276	10.05909	18
19	13.7098	13.1339	12.5933	12.0853	11.1581	10.33560	19
20	14.2124	13.5903	13.0079	12.4622	11.4699	10.59401	20
21	14.6980	14.0292	13.4047	12.8212	11.7641	10.8355	21
22	15.1671	14.4511	13.7844	13.1630	12.0416	11.0612	22
23	15.6204	14.8568	14.1478	13.4886	12.3034	11.2722	23
24	16.0584	15.2470	14.4955	13.7986	12.5504	11.4693	24
25	16.4815	15.6221	14.8282	14.0939	12.7834	11.6536	25
26	16.8904	15.9828	15.1466	14.3752	13.0032	11.8258	26
27	17.2854	16.3296	15.4513	14.6430	13.2105	11.9867	27
28	17.6670	16.6631	15.7429	14.8981	13.4062	12.1371	28
29	18.0358	16.9837	16.0219	15.1411	13.5907	12.2777	29
30	18.3920	17.2920	16.2889	15.3725	13.7648	12.4090	30
31	18.7363	17.5885	16.5444	15.5928	13.9291	12.5318	31
32	19.0689	17.8736	16.7889	15.8027	14.0840	12.6466	32
33	19.3902	18.1476	17.0229	16.0025	14.2302	12.7538	33
34	19.7007	18.4112	17.2468	16.1929	14.3681	12.8540	34
35	20.0007	18.6646	17.4610	16.3742	14.4982	12.9477	35
36	20.2905	18.9083	17.6660	16.5469	14.6210	13.0352	36
37	20.5705	19.1426	17.8622	16.7113	14.7368	13.1170	37
38	20.8411	19.3679	18.0500	16.8679	14.8460	13.1935	38
39	21.1025	19.5845	18.2297	17.0170	14.9491	13.2649	39
40	21.3551	19.7928	18.4016	17.1591	15.0463	13.3317	40
41	21.5991	19.9931	18.5661	17.2944	15.1380	13.3941	41
42	21.8349	20.1856	18.7235	17.4232	15.2245	13.4524	42
43	22.0627	20.3708	18.8742	17.5459	15.3062	13.5070	43
44	22.2828	20.5488	19.0184	17.6628	15.3832	13.5579	44
45	22.4955	20.7200	19.1563	17.7741	15.4558	13.6055	45
46	22.7009	20.8847	19.2884	17.8801	15.5244	13.6500	46
47	22.8994	21.0429	19.4147	17.9810	15.5890	13.6916	47
48	23.0912	21.1951	19.5356	18.0772	15.6500	13.7305	48
49	23.2766	21.3415	19.6513	18.1687	15.7076	13.7668	49
50	23.4556	21.4822	19.7620	18.2559	15.7619	13.8007	50
n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	7%	n

ANSWERS TO PROBLEMS

Problem Set 1-1

1. all real except $\sqrt{-2}$, $\sqrt{-4}$, which are neither positive, zero, or negative; not rational: $\sqrt{2}$, $\sqrt[3]{5}$, π, $-2\pi/3$, $\sqrt{-2}$, $\sqrt{-4}$, $\sqrt[3]{-2}$; integral: 1, 13, -11, 0, 8/4, $\sqrt[3]{-8}$; natural: 1, 13, 8/4
3. (a) and (b) rational; (c) and (d) irrational
8. $\perp$, perpendicular; $\supset$, includes

Problem Set 1-2

1. $-2 = -8/4 = \sqrt[3]{-8} = -14/7$; $2/3 = 4/6 = 0.666\ldots$
2. (a) 14, 21, 18; (b) 4, 7/3, 3/4; (c) 2, $\sqrt{3}$, 5
3. (a) yes, (b) no, (c) yes, (d) no, (e) yes, if "two" means distinct
4. (a) no, (b) no, (c) yes, (d) no, (e) no
5. $2 + 2 = 2 + (1 + 1) = (2 + 1) + 1 = 3 + 1 = 4$
6. $1 + 4 = 5$
7. (a) 5, (b) 11, (c) -5, (d) 5
8. (a) 0, (b) any x, (c) 0, (d) no x, (e) 0, (f) no x, unless $a = 0$, (g) 0
9. (a) $a = c - b$, $b = c - a$; (b) $0 = a - a$, $a = a - 0$
10. (a) $|a| + |b| = |a + b|$ if a and b are both positive or both negative, (b) $|a| + |b| > |a + b|$ if one of a and b is positive and the other negative

Problem Set 1-3

1. $d(AB) = 2$, $d(AC) = -5$, $d(CA) = 5$, $d(A'B') = -2$, $d(BA) = -2$; $AB = BA = 2$, $A'B' = 2$, $AC = 5$, $A'C = 1$
2. (a), (c), (d)
3. (a) $2 < x < 5$; (b) $-2 < x < 5$; (c) set of all points with coordinate x such that $-5 < x < 2$ or $\{x|-5 < x < 2\}$; (d) $\{x| -5 \leqq x \leqq 2\}$, $\{x|-2 \leqq x \leqq 3\}$; (e) $\{x|3 \leqq x\}$; (f) $\{x|-2 \geqq x\}$
4. (a) -13, (b) -3, (c) 3, (g) -4, (h) -10, (i) -8
5. (a) -1, (b) 2, (c) -11, (d) -2, (e) -22, (f) 38, (g) 7, (h) 0
6. (a) $a - b + c + d$, 14; (b) $a - b + c - d$, -8; (c) b, 3; (d) $-b$, -12; (e) $2a - c$, 6; (f) $-c$, -2; (g) $x + a - b + c - d$, x; (h) $x + a + b - c - d$, $x - 1$

Problem Set 1-4

1. (a) $2a - 7b - 8c$, (b) $-a + b$, (c) $-x - y - z$, (d) $2x - 12y - 11z$, (e) $-p - 4q + 3r$, (f) 0
2. (a) $3x^2 - 18x + 6$, (b) $6x^2 - 3x - 1$, (c) $3x^2 - 18xy + 6y^2$, (d) $-6x^2 + 3xy - 6y^2$

3. (a) $11a - 8b - 2c$, (b) $6a - 10b + 5c$, (c) $-4x - 8y$,
(d) $-x + 16y - 15z$, (e) $-6p - 7q + 15r$, (f) $-2p - 16q - 11r$,
(g) $-9x^2 + 7$, (h) $-11x^2 - 4x + 2$, (i) $9x^2 + y^2$,
(j) $3x^2 - 4xy - 3y^2$
4. (a) $4a - 6b + 2c$, (b) $4a - 6b + 2c$, (c) $-2a - 3b + 3c$,
(d) $-2a - 9b + 3c$, (e) $6x - 6y - 8z$, (f) $3x - 10y - 6z$

Problem Set 2-1

1. (a) $-x - 2$; (b) $-6x + 7$; (c) $2a + b$; (d) $a - 5b$; (e) $-5x - 12$;
(f) $9x - 13$; (g) $-(a + b + c)$; (h) $16a - 11b + 9c$; (i) $9a - 16b$;
(j) $8x - 5$; (k) $7a - 6c$, $x = 16$; (l) $-8a - 6c$, $x = -44$;
(m) $-4a - 4b + 3c$, $x = -22$; (n) $6a - 4b - 2c$, $y = 8$
2. (a) true; (b) false; (c) false; (d) false;
(e) false; yes, if $c = 1$ or $a = 0$
(f) false; yes, if $(a, b, c) = (3, 4, -9)$;
(g) false; no, true if $a = 0$ or if $a + b + c = 1$

Problem Set 2-2

1. (a) $-10x^3y^5$, (b) $20x^6y^7$, (c) $6125x^8y^5$, (d) $216x^5y^5$
2. (a) $3x^4 + 19x^3 + 14x^2 - 23x - 20$, (b) $-3x^3 + 26x^2 - 68x + 55$,
(c) $x^3 + 3x^2 - 4x - 12$, (d) $2x^4 - 5x^3 + 4x^2 + 5x - 12$,
(e) $a^4 + a^2 + 1$, 21, (f) $y^4 - y^2 + 2y - 1$, 77
3. (a) $10x^2 - 7xy - 12y^2$, (b) $9a^2 - 25b^2$, (c) $a^4 - b^4$,
(d) $a^5 - a^4b - ab^4 + b^5$, (e) $-2x^3y^2 + 2x^4 - 3xy^3 - y^4 + 3x^2y + xy^2$
4. (a) $9x^2 - 12xy^2 + 4y^4$, (b) $9x^2 - 4y^4$,
(c) $8x^3 + 36x^2y + 54xy^2 + 27y^3$, (d) $4x^2 + 16xy + 15y^2$,
(e) $a = 3: 3x^2 + 14x + 15$, $a = 2: 3x^2 + 11x + 10$,
$a = -2: 3x^2 - x - 10$, $a = -3: 3x^2 - 4x - 15$
5. $9x^2 + 4y^2 + 16z^2 - 12xy + 24xz - 16yz$
6. (a) $x^3 - 6x^2 + 11x - 6$, (b) $x^3 + 6x^2 + 11x + 6$, (c) $x^4 - 5x^2 + 4$

Problem Set 3-1

1. 1/3, 1/5, 3, 5/3
2. (a) use $(-x) = (-1)x$
3. If b is positive and $(1/b)$ were negative, 1 would be negative
4. (b) $-3a/5$, (c) $5x^2$, (d) $5/(xy)$, (e) $-12x/(5y)$, (f) $-17y/(29x)$,
(g) $2/(3ab)$

Problem Set 3-2

1. (a) $(y - x)/(xy)$, (b) $(9 - x^2)/(3x)$, (c) $(a^2 + b^2)/(ab)$,
(d) $(9x^3y - 3x)/y^2$, (e) $4/(x^2 - 1)$
2. (a) $-1/(xy)$, (b) -1, (c) 1, (d) $-27x^4/y^3$, (e) $-4/(x^2 - 1)$
3. (a) $-y/x$, (b) $-9/x^2$, (c) a^2/b^2, (d) $-1/(3yx^2)$, (e) $(1 - x)/(1 + x)$

4. (a) $(a^3 - 108a + 54)/(9a^2)$, (b) $(y - x + 1)/(xy)$, (c) $(10 - x^2)/(3x)$, (d) $(9x^7y - 3x^5 - 2y^4)/(x^4y^2)$, (e) $4/(x^3 - x)$
5. (a) $-8/a^2$, (b) $-1/(x^2y^2)$, (c) $-1/(3x)$, (d) $54/y$, (e) $-16/(x^3 - x)$
6. (a) $-1/18$, (b) -1, (c) $-3x$, (d) $27x^8/(2y^5)$, (e) $-x/(x^2 - 1)$
7. (a) $(a^2 - b^2)/(a^2 + b^2)$, (b) $(2x + y)/y$, (c) -1, (d) $(1 - a)/(a^2 + a)$, (e) $(xy^2 - xy + y^2)/(x^2y - xy + x^2)$

Problem Set 3-3

1. (a) 8/125, (b) 62/25, (c) 27/40, (d) 76/25
2. (a) 0.78125, (b) 0.0144, (c) 1.825, (d) 0.35625
5. (a) 427/1000; (b) 21/100; (c) 5/9, 16/45, 196/45; (d) 19/33, 151/330, 1141/330; (e) 4352/999
6. (a) 0.545454, 0.5455; (b) 0.351351, 0.3514; (c) 0.307692, 0.3077; (d) 0.384615, 0.3846; (e) 0.428571, 0.4286; (f) 0.176471, 0.1765
7. (a) 8.506, (b) 8.314, (c) 8.5398, (d) 1.1557, (e) 0.9731
8. 84.791
9. (a) 15.10, (b) 33.47, (c) 37.50, (d) 0.2617

Problem Set 3-4

1. \$14.20
2. \$37.10
3. \$361.01
4. \$29.28, 13%
5. (a) 6.2%, (b) 115.5%, (c) 0.758%
6. (a) 25%, (b) \$3.68, (c) $21\frac{1}{3}$%, 19.05%, 16%
7. 2% gain
8. (a) \$12.79, (b) 18.81%
9. (a) \$102.60, $35\frac{7}{8}$%, (b) gain \$1.40
10. 7.2921%
11. (a) \$1625, (b) \$2397.60, (c) \$3566.00
12. (a) 69.6, (b) 73.4, (c) 11000
13. \$16.18, (\$16)
14. \$1768
15. (a) \$89.29, (b) \$125
16. \$53300
17. \$17.64, less

Problem Set 3-5

1. $(2x - 3y)(2x + 3y)$
2. $(a - 3b)^2$
3. $(2x + 3y)^2$
4. $(\frac{2}{3}x - \frac{3}{2}y)^2$
5. $3(3x - 4y)(3x + 4y)$
6. $2a(a - 2b)^2$
7. $(x - 3y)(x^2 + 3xy + 9y^2)$
8. $(x + 3y)^3$
9. $(x - 2y)(x + 2y)(x^2 + 2xy + 4y^2)(x^2 - 2xy + 4y^2)$
10. $(x^2 + 4y^2)(x^4 - 4x^2y^2 + 16y^4)$
11. $(a - 2x)(3b - 4y)$
12. impossible
13. $(5a + 6c)(a - 3b)$
14. impossible
15. $(x - 3)(3x - 2)$
16. $(x + 3)(3x + 2)/3$
17. $(x - 4/3)(x - 2/3)$
18. $(3x - 4)(x + 2)$
19. impossible

20. $(3x + 2)(5x + 4)$
21. $(3a - 4b)(4a + 5b)$
22. $(4a - 5b)(3a - 5b)$
23. $(3x + 7y)(5x - 2y)$
24. impossible
26. yes, no

Problem Set 3-6

1. (a) $x^3 - 6x - 4 = (x - 2)(x^2 + 2x - 2) - 8$
   (b) $x^3 - 6x - 4 = (x^2 + x - 2)(x - 1) - (3x + 6)$
   (c) $x^3 - 12x - 16 = (x + 2)(x^2 - 2x - 8)$
   (d) $x^3 - 12x - 16 = (x^2 - 2x - 8)(x + 2)$
   (e) $x^3 - 12x - 16 = [(2x - 3)(4x^2 + 6x - 39) - 245]/8$
   (f) $F(x) = (x^2 + 2x + 3)(x^2 + x + 2)$
   (g) $F(x) = (x^2 - 2x + 3)(x^2 + 5x + 14) + 20x - 36$
2. (a) $-12, 0, 0$; (c) $(x + 1)(x - 2)(x^2 - 2x + 7)$
3. (a) $-16, 4, 180, 0$; (b) 0; (c) $x + 3, x + 3, x^2 - x - 1$
6. (a) $a^2 + ab + b^2$; (b) $a^2 - ab + b^2$;
   (c) $F(a, b) = (a + b)(a^2 - 4ab + 7b^2) - 8b^3$, $F(-b, b) = -8b^3$;
   (d) $x^2 + y^2$, $F(-y, y) = 0$;
   (e) $F(x, y) = (x + y)(x^2 - 2xy + 3y^2) - 4y^3$, $F(-y, y) = -4y^3$

Problem Set 3-7

1. $F(3) = 0$, $(x - 3)(3x - 2)$
2. $F(\frac{2}{3}) = 0$, $(3x - 4)(3x - 2)$
3. $F(a) \neq 0$ for $a = \pm 8, \pm 4, \pm 2, \pm 1, \pm\frac{8}{3}, \pm\frac{4}{3}, \pm\frac{2}{3}, \pm\frac{1}{3}$; impossible
4. $F(-\frac{2}{3}) = 0$, $(3x + 2)(5x + 4)$
5. $F(a) \neq 0$ for $a = \pm 1, \pm 2, \pm 4$; impossible
6. $a > 0$; $F(a) \neq 0$ for $a = 1, 2, 4, 8, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}$; impossible
7. $F(-y, y) = 0$, $(x + y)(x^2 - xy + y^2)$
8. $F(\pm y, y) \neq 0$, impossible
9. $F(3y, y) = 0$, $(x - 3y)(x^2 + 3xy + 9y^2)$
10. $F(9y/4, y) = 0$, $(4x - 9y)^2$
11. $F(4b/3, b) = 0$, $(3a - 4b)(4a + 5b)$
12. $F(5b/3, b) = 0$, $(4a - 5b)(3a - 5b)$
13. $F(1) = 0 = F(2) = F(3)$, $(x - 1)(x - 2)(x - 3)$
14. $F(3) = 0$, $F(x) = (x - 3)(x^2 - 3x - 2)$, $Q(a) \neq 0$ for $a = \pm 1, \pm 2$
15. $F(1) \neq 0$, $F(-1) = 0$, $(x + 1)(x^2 - x - 1)$
16. $F(1) \neq 0$, $F(-1) \neq 0$, impossible
17. $F(-2) = 0 = F(-3)$, $(x + 2)^2(x + 3)$
18. $F(3) = 0$, $(x - 3)^3$
19. x negative, $F(-1) = 0 = F(-3)$, $(x + 1)^3(x + 3)$
20. $F(1) = 0 = F(-3)$, $(x - 1)^2(x + 3)^2$
21. $F(3) = 0$, $F(1) \neq 0$, $F(-1) \neq 0$, $(x - 3)(x^3 + 2x + 1)$
22. $F(-2) = 0$, $Q(-2) = 0$, $(x + 2)^2(x^2 - 2x - 2)$
23. $(x - 5)/(2x - 5)$
24. $(x - 5)/(x + 5)$

25. $(2a - b)/(a + b)$ 26. $(a + b)/(a - b)$
27. $(a^2 - ab + b^2)/(a - b)$
28. $(x^2 + xy + y^2)/(x^2y + xy^2)$, $xy \neq 0$
29. $8/(x^3 - x^2 - x + 1)$ 30. $4x/(x^2 - 4)$
31. $2/(2x^4 + 11x^3 + 16x^2 + x - 6)$; $x = 1$, $F(x) = \frac{1}{12}$

Problem Set 4-1

1. (a) no, (b) yes, (c) yes, (d) no
2. (a) no, (b) yes, (c) yes, neither equation has a solution
3. (a) $\frac{8}{9}$, (b) $\frac{26}{15}$, (c) $\frac{4}{3}$, (d) $\frac{48}{5}$, (e) $-\frac{4}{15}$
4. (a) $\frac{5}{2}$, (b) 3, (c) 0, (d) $\frac{3}{2}$
5. (a) 12; (d) -8; (e) 6, (b), (c), (f) no solution
6. (a) $x = yz/(y + z)$, $y = xz/(z - x)$; (b) $h = 3V/(\pi r^2)$;
   (c) $n = (A - P)/(Pi)$; (d) $d = (P - A)/(nP)$;
   (e) $y = (b/a)(a - x)$; (f) $x = 8 - p/5$; (g) $x = (8 - 4p)/p$;
   (h) $y = (-A/B)x - (C/B)$, $x = (-B/A)y - (C/A)$; (i) $9C/5 + 32$

Problem Set 4-2

1. 11, 7 2. 11, 7
3. (a) 9, 10, 11, 12; (b) $A - 2$ a multiple of 4
4. (a) $A/3 - 2$, $A/3$, $A/3 + 2$, where A is odd and a multiple of 3; (b) 105
5. $\frac{25}{4}$, $\frac{15}{4}$ 6. (b) 11, 9 (c) 104, 27, 25
7. (a) $AC = 4\frac{1}{8}$ cm, (b) $AC = 3\frac{5}{8}$ cm
8. (a) $4\frac{3}{8}$, (b) $3\frac{3}{4}$, 5
9. (a) $AC = 7.5$, (b) no solution; the assumption that C is on the segment AB leads to a point that is not on this segment
10. (a) $5\frac{1}{2}$ in., $3\frac{1}{2}$ in., (b) $6\frac{1}{2}$ in., $2\frac{1}{2}$ in.
11. 2 in. 12. 3 ft by 9 ft
13. \$3500, \$3875, \$4625 14. \$3200, \$3200, \$4200, \$9400
15. 166, 334 16. \$8825, \$8825, \$7350, approx.
17. 10.7 gal 18. 2 gal, 3 gal
19. 30, 20 lb 20. $\frac{25}{3}$ lb
21. (a) 6.55 ft from lighter boy, (b) 6.5 ft from lighter boy
22. 1.75 ft in front of A 23. 7.6 ft from A
24. $6\frac{2}{7}$ ft from A; if x is the distance from the center to the fulcrum and you assume the center is on A's side of the board, x will turn out to be negative
25. 48 min 26. A travels 140 mi in 4 hr
27. F travels 420 mi in 12 hr 28. $5\frac{5}{11}$ min after 1
29. (a) A runs 10 and B runs 9 laps in 8.8 min,
   (b) A runs $\frac{10}{19}$ and B runs $\frac{9}{19}$ of the lap in $\frac{44}{95}$ min
30. 3.24 mph
31. (a) $\frac{25}{11}$ sec, (b) $\frac{65}{11}$ sec, (c) $\frac{90}{11}$ sec, (d) $\frac{18}{11}$ sec
32. \$31 approx.

Problem Set 4–3

1. (a) 46%, (b) 40.5%, (c) 49.6%, (d) 51.4%
2. 22%, 22.29%
3. $r = (24 - 14000/B)\%$, 21.2%
4. $r = (30 - 58000/B)\%$, 23.56%
5. 27.78 mph
6. (a) $r > 20$ mph, (b) late, (c) on time

Problem Set 4–4

1. ordinary: (a) \$5.00, (b) \$5.56, (c) \$5.83, (d) \$6.00;
   exact: (a) \$5.04, (b) \$5.59, (c) \$5.86, (d) \$6.02
2. (a) \$98.28, (b) \$102.89, (c) \$933.44, (d) \$1059.27
3. $d = 6.67\%$, $i = 6.90\%$
5. $d = 7.5\%$, $i = 7.89\%$
6. (b) \$1655.10
7. \$579.38
8. $10\frac{2}{3}\%$
9. \$31.58, \$33.33

Problem Set 4–5

1. (a) $f(2 + y) = 3y + 2$, (b) $f(2 + y) = (3y + 2)/(y + 4)$,
   (c) $f(2 + y) = (2y + 4)/(y^2 + 4y + 5)$, (d) $f(2 + y) = y^2 + 2y + 4$,
   $f(x + k) = x^2 + (2k - 2)x + (k^2 - 2k + 4)$
2. (a) domain and range: all reals; (b) range: $f(x) \geqq 2$;
   (c) domain: $x > -\frac{2}{3}$
3. (a) $f(x) \geqq -2$, (b) domain: $x \geqq \frac{2}{3}$, (c) domain: $x \geqq \frac{2}{3}$, range: $y \geqq 0$
4. (a) $x - 3y + 7 = 0, \frac{7}{3}$; (b) $3x + y - 9 = 0, 9$;
   (c) $x + y - 1 = 0, 1$; (d) $2x - y = 0, 0$
5. (a) $y = 2x + 4, -2$; (b) $5x - 2y = 5, 1$; (c) $3x + 2y = 20, \frac{20}{3}$;
   (d) $2x + 3y + 6 = 0, -3$
6. use the points (a) $(0, 3), (-2, 0), (2, 6)$; (b) $(0, 0), (5, 2), (-5, -2)$;
   (c) $(0, 3), (\frac{9}{2}, 0), (3, 1)$; (d) $(0, -6), (\frac{18}{5}, 0), (3, -1)$; (e) line segments joining $(2, 1)$ to $(0, 5)$ and to $(4, 1)$; (f) line segments joining $(3, 3)$ to $(0, -3)$ and to $(3, 6)$
7. (a) $(0, 6), (8, 0)$; (b) $(0, 0), (4, 3)$; (c) $(0, -3), (4, 0)$;
   (d) $(0, \frac{3}{2}), (\frac{12}{5}, 0)$; (e) $(0, -1), (-1, 0)$; (f) $(0, 32), (-20, 0)$

Problem Set 4–6

1. $d = 25t$, $(0 \leqq t \leqq 1.6)$; $d = 40$, $(1.6 \leqq t \leqq 2.1)$; $d = 40t - 44$, $(2.1 \leqq t \leqq 3.6)$; 27.78 mph; slope of segment joining O to final point.
2. 20 mph, $d = 40t$, $(0 \leqq t \leqq \frac{3}{4})$; $d = 20t + 15$, $(\frac{3}{4} \leqq t \leqq \frac{3}{2})$
3. $(F = 32, C = 0)$, $(F = 212, C = 100)$
4. $P = 1 - 0.025T$; use points $(0, 1), (12, 0.7)$
5. (a) 45, (b) 9, (c) line segments joining $(7, 10)$ to $(0, 45)$ and to $(10, 10)$
6. (a) 16, 64; (b) line segments: $p = 16 - x/4$, $(0 \leqq x \leqq 48)$; $p = 4$, $(48 \leqq x \leqq 80)$
7. (a) use points $(0, \frac{16}{3})$, $(12, 0)$; (b) $p = (\frac{16}{3}) - 4x/9$, $m = -\frac{4}{9}$, $(\frac{4}{3}, 9)$

8. $(-10, 0)$
9. use points $(0, 3)$, $(-4, 0)$ and $(8, 9)$ as a check, $(0 \leqq x \leqq 8)$
10. line segments joining $(4, 9)$ to $(0, 3)$ and to $(9, \frac{37}{3})$
11. $Q = 5 + 0.1x$, $(40 \leqq x \leqq 100)$; $Q = 4 + 0.11x$, $(0 \leqq x \leqq 100)$
12. $4x + 7y = 35$, $m = -\frac{4}{7}$, $(7, 1)$
13. $Ax + By + Cz = E$; part of the plane in positive octant; consider problem algebraically

Problem Set 5-1

1. (a) $(3, 5)$; (b) $(2, 10)$; (c) $(5, -2)$; (d) $(-1, -2)$; (e) $(-6, 7.5)$; (f) $(\frac{11}{8}, \frac{3}{2})$; (g) (any x, $\frac{2}{3}x + 4$); (h) no solution
3. (a) $(-2, -3)$; (b) $(\frac{14}{5}, \frac{6}{5})$; (c) $(4, 0)$; (d) $(\frac{4}{3}, -1)$; (e) $(2.3, 1.1)$; (f) $(-1, \frac{3}{2})$
4. (a) inconsistent, parallel lines; (b) dependent, same line; (c) inconsistent; (d) dependent; (e) inconsistent; (f) inconsistent

Problem Set 5-2

1. (a) $3x - 4y + 15 = 0$, $\frac{15}{4}$; (b) $2x + 3y = 16$, $\frac{16}{3}$; (c) $2x - 3y + 21 = 0$, 7; (d) $2x + 5y + 19 = 0$, $-\frac{19}{5}$
2. (a) $m = 1$, $y = x + 3$; (b) $m = -\frac{7}{2}$, $7x + 2y = 6$; (c) $m = -\frac{11}{6}$, $11x + 6y + 4 = 0$; (d) $m = 0$, $y = -2$
3. (a) $5x - 3y = 1$; (b) $3x + 4y = 17$, (c) $x + y = 2$, (d) $7x - 2y = 20$, (e) $4x + 3y = 1$, (f) $3x - 5y = 11$
4. $(\frac{15}{2}, \frac{5}{2})$
5. $(7, -2)$
6. $AC = \frac{37}{6}$ in., $CB = \frac{7}{3}$ in.
7. $AC = \frac{54}{7}$ in.
8. \$1800 at 5%, \$1200 at 8%
9. (a) $\frac{2}{3}$ at 4%, $\frac{1}{3}$ at 8%; (b) $\frac{11}{31}$ at 4%, $\frac{20}{31}$ at 16%
10. S = \$3000, W = \$5000
11. (a) $F = -40$, (b) $F = 320$, (c) $F = 185$
12. $\frac{8}{3}$ hr, $\frac{400}{3}$ mi from A
13. 252 mi, 7 hr 12 min after freight starts
14. 26, 36.5 lb approx.
15. $\frac{600}{7}$, $\frac{450}{7}$ lb
16. 8, $\frac{72}{7}$ oz
17. $\frac{45}{11}$ gal milk, $\frac{10}{11}$ gal cream
18. 30 lb at 85¢, 20 lb at 60¢
19. 3.6 qt of 45% alcohol, 2.4 qt of 70% alcohol
20. 91¢, 68¢ a lb

Problem Set 5-3

1. (a) $(\frac{20}{9}, \frac{16}{3})$, (b) $(2, 6)$, $p_1 - p = \frac{2}{3}$, $x - x_1 = \frac{2}{9}$
2. (a) $p_1 = \frac{16}{3} + 2t/3$, (b) $\frac{9}{4}$
3. (a) $(3, 21)$; (b) $(\frac{31}{9}, \frac{169}{9})$, $p - p_1 = \frac{20}{9}$, $x_1 - x = \frac{1}{9}$
4. $\frac{5}{9}$
5. (a) $(8, \frac{14}{3})$, (b) $(6, 6)$
6. (a) $(8 - t, (14 + 2t)/3)$; (b) $p - p_2 = \frac{1}{3}$, $x_2 - x = 2$
7. $t = 2$
8. (a) $(4, \frac{10}{3})$, (b) $(\frac{26}{7}, \frac{18}{7})$
9. $(\frac{5}{4}, \frac{3}{2})$, $(\frac{53}{52}, \frac{24}{13})$
10. $p_1 = \frac{3}{2} + 9t/13$

Problem Set 5–4

1. 1
2. -1
3. -18
4. 4
5. 0
6. (a) -1, (b) -18, (c) 0
7. elements of diagonal are 1, 1, -1
8. elements of diagonal are 1, 1, 0
10. $a_1C_1 + a_2C_2 + a_3C_3 = 0$, $a_1A_2 + b_1B_2 + c_1C_2 = 0$, etc.
11. (a) $(\frac{5}{3}, -\frac{2}{3}, \frac{1}{3})$, (b) $(3, -2, 1)$, (c) $(\frac{7}{2}, 3, \frac{5}{2})$, (d) $(\frac{4}{15}, \frac{20}{15}, \frac{2}{15})$
12. $(3, -1, -2)$
13. (d) $D = -30$, $D_1 = -8$, etc.

Problem Set 5–5

1. use traces in two coordinate planes
2. use the intercepts
3. (a) $4x + 3y + 5z = 100$, (b) $(\frac{25}{2}, \frac{25}{4}, \frac{25}{4})$
4. (a) $6x + 2y + 10z = 120$, (b) $(\frac{60}{11}, \frac{180}{11}, \frac{60}{11})$
5. boundary: (a) triangle, (b) trapezoid, (c) triangle, (d) trapezoid, (e) triangle, (f) trapezoid
6. $x = (10 - p + q)/2$, $y = (20 - p - q)/2$
7. (a) $x = 18 - 3p + 2q$, $y = 3 + p - q$;
   (b) $x = 4 - 2p + q$, $y = 20 + p - 5q$;
   (c) $x = 4 - 10p + 7q$, $y = 3 + 7p - 5q$
8. (a) $D = 0$, $D_1 \neq 0$, first and third planes are parallel;
   (b) $D = 0$, $D_1 \neq 0$, second and third planes are parallel;
   (c) $D = D_1 = D_2 = D_3 = 0$, parallel planes;
   (d), (e) and (f) $D = 0$, $D_1 \neq 0$, no two planes are parallel, line of intersection of two planes is parallel to the third plane
9. (a) $D = D_1 = D_2 = D_3 = 0$, first and third planes are identical; the planes have a line in common; $(24 + z, -12, -2z)$;
   (b) $D = D_1 = D_2 = D_3 = 0$, distinct and nonparallel planes with a common line; $(24 + z, -12 - 2z, z)$;
   (c) description as in (b); $([1 + 2z]\ 3, [-1 + 4z]/9, z)$;
   (d) description as in (b); $(18 - 51z\ 7, 2z\ 7, z)$
10. (a) $D = -5 \neq 0$, (b) $D = -2 \neq 0$
11. (a) $D = 0$, $(-5z\ 3, 2z\ 3, z)$, all points on the line joining $(0, 0, 0)$ to $(-5, 2, 3)$; (b) $D = 0$, $(-17z\ 7, 2z\ 7, z)$, all points on the line joining $(0, 0, 0)$ to $(-17, 2, 7)$
12. (a) common point $(\frac{1}{2}, \frac{3}{2})$; (b) triangle $(\frac{1}{2}, \frac{5}{2})$, $(-\frac{11}{2}, -\frac{15}{2})$, $(\frac{19}{6}, \frac{7}{6})$;
   (c) triangle $(-1, -2)$, $(-\frac{1}{7}, -\frac{18}{7})$, $(-2, -\frac{7}{2})$;
   (d) common point $(-1, -2)$, (e) common point $(-6, \frac{15}{2})$;
   (f) triangle $(-6, \frac{15}{2})$, $(\frac{78}{17}, \frac{105}{34})$, $(\frac{18}{7}, \frac{15}{14})$

Problem Set 6–1

2. (a) 0.6245, (b) 1.975, (c) 0.1975, (d) 62.45
3. (a) 3.6, (b) 1.1, (c) 4.4, (d) 1.4, (e) 6.1, (f) 1.9

4. (a) 4.796, (b) 15.17, (c) 23.73, (d) 0.7503, (e) 2.912, (f) 4.870
5. (a) 4.79583, (b) 15.1658, (c) 23.7276, (d) 0.750333, (e) 2.91187, (f) 4.87032
6. (a) 1.773, (b) 5.605, (c) 0.5642, (d) 1.649, (e) 5.213, (f) 0.6065
7. (a) 2.376, (b) 2.210, (c) 0.640
8. (a) 1.844, (b) 1.309, (c) 2.483, (d) 1.773
10. consider the equation $\frac{2}{3} = p^2/q^2$, p and q relatively prime integers, etc.

Problem Set 6-2

1. (a) $0, -\frac{5}{2}$; (b) $0, \frac{2}{5}$; (c) $-\frac{1}{2}, -\frac{1}{2}$; (d) $-3, -3$; (e) $-2, -\frac{14}{5}$; (f) $-\frac{1}{2}, 1$; (g) $-1 \pm \sqrt{7}$; (h) $-4 \pm \sqrt{14}$; (i) $(-4 \pm \sqrt{13})/3$ or $-2.535, -0.131$; (j) $(-6 \pm \sqrt{56})/3$ or $-2.697, 0.297$
2. (a) complex; (b) $-5.317, 1.317$; (c) $-2.351, 0.851$; (d) complex; (e) $-0.180, 1.847$; (f) $0.232, 1.434$; (g) complex; (h) $-0.458, 2.124$
3. (a) $-\frac{1}{2}, 1$; (b) $(-4 \pm \sqrt{13})/3$; (c) $(-3 \pm \sqrt{41})/4$; (d) complex; (e) complex; (f) $(5 \pm \sqrt{61})/6$
4. (a) none; (b) 0; (c) $\frac{5}{2}$; (d) $-1, 5$; (e) 7; (f) $6 - \sqrt{57} = -1.550$; (g) $(d - b)/(a - c)$
5. (a) 2, (b) -5, (c) $2 \pm \sqrt{2}$, (d) $-1 \pm \sqrt{2}$, (e) -2, (f) none
6. (a) 5; (b) 40; (c) $-1, 3$; (d) $4, 20$; (e) 340; (f) $\frac{1}{2}$
7. (a) $1, 7$; (b) none; (c) $1, -5$; (d) $-3, \frac{5}{2}$; (e) $c > 0$, $(b + c)/a$, $(b - c)/a$; $c < 0$, none
8. (a) 3; (b) $-2, \frac{10}{3}$; (c) $-\frac{1}{3}, 7$; (d) $\frac{14}{3}, 10$; (e) $(a + b)/2$; if $a = b$, any x
9. (a) $(a + b)/2$; (b) $(a - kb)/(1 - k)$, $(a + kb)/(1 + k)$ which correspond to distinct points for $k > 0$, $k \neq 1$, $a \neq b$
10. (a) ± 3; (b) $-2 \leqq x \leqq 2$; (c) none; (d) $\frac{3}{2}$; (e) $-2, 10$; (f) $0 \leqq x \leqq 8$; (g) none; (h) 1

Problem Set 6-3

1. (a) use $(0, 0)$, $(6, 0)$, $V(3, -9)$; (b) use $(0, 0)$, $(4, 0)$, $V(2, 4)$; (c) use $(0, 0)$, $(-4, 0)$, $V(-2, -4)$; (d) use $V(0, 9)$, $(\pm\frac{3}{2}, 0)$; (e) use $V(0, 9)$, $(\pm 1, 13)$, $(\pm 2, 25)$; (f) use $V(0, 6)$, $(\pm\sqrt{2}, 0)$, $(\pm 2, 6)$; (g) use $(0, 16)$, $V(4, 0)$, $(8, 16)$; (h) use $(0, 1)$, $V(-2, 0)$, $(-4, 1)$, $(1, \frac{3}{4})$, $(-3, \frac{9}{4})$
2. (a) use $(0, -5)$, $(-1, 0)$, $(5, 0)$, $V(2, -9)$, $(4, -5)$, $(-2, 7)$, $(6, 7)$; (b) use $(0, -4)$, $(1, 0)$, $(4, 0)$, $V(\frac{5}{2}, \frac{9}{4})$, $(5 \ -4)$; (c) use $(0, 9)$, $(-\frac{3}{2}, 0)$, $(3, 0)$, $V(\frac{3}{4}, \frac{81}{8})$, $(\frac{3}{2}, 9)$; (d) use $(0, 27)$, $(-\frac{9}{2}, 0)$, $(\frac{3}{2}, 0)$, $V(-\frac{3}{2}, 36)$, $(-3, 27)$; (e) use $(0, 7)$, $V(\frac{5}{2}, \frac{3}{4})$, $(5, 7)$, $(1, 3)$, $(4, 3)$; (f) use $(0, \frac{3}{2})$, $V(-2, \frac{1}{2})$, $(-4, \frac{3}{2})$, $(2, \frac{9}{2})$, $(-6, \frac{9}{2})$; (g) use $(0, -6)$, $(-4.4, 0)$, $(1.4, 0)$, $V(-\frac{3}{2}, -\frac{33}{4})$, $(-3, -6)$; (h) use $(0, -6)$, $(3 \pm \sqrt{3}, 0)$, $V(3, 3)$, $(6, -6)$;
3. (a) use $(0, 0)$, $(0, 4)$, $V(-4, 2)$, $(12, 6)$, $(12, -2)$; (b) use $(0, 0)$, $(0, -4)$, $V(-4, -2)$, $(12, -6)$, $(12, 2)$; (c) use $(0, 0)$, $(0, 2)$, $V(6, 1)$, $(-18, 3)$, $(-18, -1)$;

(d) use $V(0, 2)$, $I(4, 0)$, $(9, -1)$, and from symmetry $(4, 4)$, $(9, 5)$;
(e) use $V(0, 8)$, $I(8, 0)$, $(13, -1)$, and from symmetry $(8, 4)$, $(13, 5)$;
(f) use $V(-2, 2)$, $I(0, 2 \pm \sqrt{2})$, $I(2, 0)$, $(7, -1)$ and from symmetry; $(2, 4)$, $(7, 5)$

4. (a) $y \geqq 0$; use $V(-2, 0)$, $(0, \sqrt{8})$, $(2, 4)$, $(7, 6)$;
(b) first quadrant; use $V(-3, 0)$, $(0, \sqrt{6})$, $(5, 4)$, $(15, 6)$;
(c) first quadrant; use $V(\frac{9}{2}, 0)$, $I(0, 3)$, $(\frac{5}{2}, 2)$;
(d) above line $y = -1$, between $x = 0$ and $x = \frac{9}{4}$; use $V(\frac{9}{4}, -1)$, $I(2, 0)$, $I(0, 2)$, $(1, \sqrt{5} - 1)$ or $(1, \frac{5}{4})$
5. $y = x^2 - 4x + 4$, $V(2, 0)$; $x = (y^2 - 7y + 12)/6$, $V(-\frac{1}{24}, \frac{7}{2})$
6. (a) $y = x^2 - 2x + 3$, $V(1, 2)$, $I(0, 3)$; $x = (-y^2 + 11y - 12)/6$, $I(-2, 0)$, $I(0, 1.2)$, $I(0, 9.8)$, $V(\frac{73}{24}, \frac{11}{2})$; (b) $y = (x - 3)^2$, $V(3, 0)$; $x = (y^2 - 19y + 90)/30$, $I(0, 10)$, $V(-\frac{1}{120}, \frac{19}{2})$;
(c) $y = (x^2 - 5x + 6)/2$, $V(\frac{5}{2}, -\frac{1}{8})$; symmetry shows $x = (y^2 - 5y + 6)/2$, $V(-\frac{1}{8}, \frac{5}{2})$; (d) $y = x^2/4 + 1$, $V(0, 1)$; $x = -(y^2 - 9y + 8)/3$, $V(\frac{49}{12}, \frac{9}{2})$, $I(-\frac{8}{3}, 0)$
7. $y = \sqrt{9 + 4x}$, $I(0, 3)$, $V(-\frac{9}{4}, 0)$, $y \geqq 0$
8. $x = y^2 + 2y - 8$; $y = \sqrt{(17 + 4x)/5}$, $V(-\frac{17}{4}, 0)$, $I(0, \sqrt{\frac{17}{5}})$, good agreement in first quadrant only
9. $y = \sqrt{32 - 8x}$, $I(0, \sqrt{32})$, $V(4, 0)$
10. (b) $a = D_1/D$, $D_1 = 0$, $D \neq 0$; (c) $a = D_1/D$, $D_1 = 0$, $D = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \neq 0$

Problem Set 6–4

1. $b^2/(4m)$
2. (a) use $(0, 0)$, $(9, 0)$, $V(\frac{9}{2}, 81)$; (b) use $(0, 0)$, $(\frac{27}{2}, 0)$, $V(\frac{27}{4}, \frac{243}{8})$, $(3, 21)$, $(9, 27)$; (c) use $(0, 0)$, $(105, 0)$, $V(52.5, 919)$, $(24, 648)$, $(72, 792)$; (d) use $(0, 0)$, $(\frac{12}{5}, 0)$, $V(\frac{6}{5}, \frac{12}{5})$
3. use as (p, R): (a) $(0, 0)$, $(20, 0)$, $V(10, 500)$, $(5, 375)$, $(15, 375)$;
(b) $(0, 0)$, $(10, 0)$, $V(5, 15)$, $(2, \frac{48}{5})$, $(8, \frac{48}{5})$
4. (a) use $V(0, 48)$, $(4, 0)$, $(2, 36)$; (b) use $V(0, 20)$, $(\sqrt{5}, 0)$, $(\frac{3}{2}, 11)$;
(c) use $V(-1, 48)$, $(0, 45)$, $(3, 0)$, $(1, 36)$, $(2, 21)$;
(d) use $V(-1, 20)$, $(0, 16)$, $(-1 + \sqrt{5}, 0)$, $(1, 4)$ with large x-unit;
(e) use $0 \leqq x \leqq 5$, $(0, 25)$, $V(5, 0)$, $(2, 9)$, $(8, 9)$;
(f) use $0 \leqq x \leqq 3$, $(0, 15)$, $(3, 0)$, $(5, 0)$, $V(4, -1)$, $(8, 15)$;
(g) use $0 \leqq p \leqq 6$, $(0, 6)$, $V(36, 0)$, $(20, 4)$;
(h) use $0 \leqq p \leqq 4$, $(0, 4)$, $(0, -8)$, $(32, 0)$, $V(36, -2)$, $(20, 2)$;
(i) use $0 \leqq p \leqq 3$, $(0, 3)$, $V(\frac{9}{2}, 0)$, $(\frac{5}{2}, 2)$;
(j) use $0 \leqq p \leqq 4$, $(0, 4)$, $V(\frac{16}{3}, 0)$, $(4, 2)$;
5. (a) use part in first quadrant, $V(-4, 0)$, $(0, \pm 4)$, $(5, 6)$, $(8, \sqrt{48} \doteq 7)$;
(b) use part in first quadrant, $(0, 0)$, $(0, 4)$, $V(-1, 2)$, $(3, 6)$, $(8, 8)$;
(c) use part in first quadrant, $(0, 3)$, $V(-\frac{9}{2}, 0)$, $(\frac{7}{2}, 4)$, $(8, 5)$;
(d) use part in first quadrant, $(0, 2)$, $(0, -4)$, $(-4, 0)$, $V(-1, -\frac{9}{2})$, $(8, 4)$, $(\frac{7}{2}, 3)$; (e) use part in first quadrant, $(0, \sqrt{8})$, $V(-8, 0)$, $(8, 4)$, $(4, \sqrt{12})$;
(f) use part in first quadrant, $(0, 3)$, $V(-\frac{9}{4}, 0)$, $(4, 5)$, $(10, 7)$;
(g) use $(0, 5)$, $(5, 7)$, $(10, 8)$, $V(\frac{25}{2}, \frac{65}{8})$, $(x \leqq 10)$;

(h) use for Q_1: $0 \leqq x \leqq 4$, $(0, 0)$, $(2, \frac{7}{4})$, $(4, 4)$, $V(-6, -\frac{9}{4})$; use for Q_2: $4 \leqq x \leqq 12$, $V(2.4, 0)$, $(4, 4)$, $(6, 6)$, $(8.8, 8)$, $(12, 9.8)$

Problem Set 6-5

1. (a) circle, center $(0, 0)$, $r = 4$; (b) $V(\pm 4, 0)$, $V(0, \pm 3)$
   (c) $V(\pm 4, 0)$, $V(0, \pm 5)$; (d) $V(\pm\sqrt{2}/2, 0)$, $V(0, \pm 1)$
2. (a) $V(\pm 1, 0)$, asymptotes $y = \pm x$;
   (b) $V(0, \pm 1)$, asymptotes $y = \pm x$;
   (c) $V(\pm 4, 0)$, asymptotes $y = \pm(\frac{3}{4})x$;
   (d) $V(\pm 4, 0)$, asymptotes $y = \pm(\frac{5}{4})x$;
   (e) $V(0, \pm 3)$, asymptotes $y = \pm(\frac{3}{4})x$;
   (f) $V(0, \pm 1)$, asymptotes $y = \pm\sqrt{2}\,x$
3. (a) asymptotes $x = 0$, $y = 0$; use $(2, \frac{3}{2})$, $(3, 3)$, $(\frac{3}{2}, 2)$ and symmetry for points in third quadrant; (b) asymptotes $x = 0$, $y = 0$; use $(1, -\frac{3}{4})$, $(\frac{3}{2}, -\frac{3}{2})$, $(\frac{9}{4}, -1)$ and symmetry for points in third quadrant; (c) asymptotes $x = 2$, $y = 0$; use $(0, -1)$, $(1, -2)$, $(3, 2)$, $(4, 1)$; (d) asymptotes $x = 2$, $y = 0$; use $(0, 2)$, $(1, 4)$, $(3, -4)$, $(4, -2)$; (e) asymptotes $x = -2$, $y = 0$; use $(-1, 4)$, $(0, 2)$, $(2, 1)$ and points obtained from symmetry with regard to $(-2, 0)$: $(-3, -4)$, $(-4, -2)$, $(-6, -1)$; (f) asymptotes $x = 4$, $y = -1$; use $(0, 0)$, $(2, 1)$, $(3, 3)$ and points obtained from symmetry with regard to $(4, -1)$ including $(8, -2)$
4. (a) circle, $C(-2, 3)$, $r = \sqrt{13}$; check: $x = 0$, $y = 0, 6$; $y = 0$, $x = 0, -4$; (b) circle, $C(\frac{3}{2}, -\frac{1}{2})$, $r = \frac{3}{2}$; check: $x = 0$, $y = -\frac{1}{2}$; $y = 0$, $x = 0.1, 2.9$ approx.; (c) ellipse, $C(2, -2)$, x_1-intercept ± 2, y_1-intercept ± 1; check: $x = 0$, $y = -2$; $y = 0$, x not real; (d) ellipse, $C(2, 3)$, x_1-intercept $\sqrt{2}$, y_1-intercept $\sqrt{3}$; check: $x = 0$, y not real; $y = 0$, x not real; (e) parabola $y_1 = -4x_1^2$, $O_1(-\frac{3}{2}, 36)$, $V(x_1 = 0, y_1 = 0)$, $x_1 = \pm 3$, $y_1 = -36$; check: $x = 0$, $y = 27$; $y = 0$, $x = \frac{3}{2}, -\frac{9}{2}$; (f) parabola $y_1^2 = x_1$, $O_1(2, 2)$, $V(x_1 = 0, y_1 = 0)$, $(x_1 = 4, y_1 = \pm 2)$; check: $y = 0$, $x = 6$; $x = 0$, y not real; (g) hyperbola $x_1^2 - y_1^2 = 4$, $O_1(-2, -3)$, asymptotes $y_1 = \pm x_1$, $V(x_1 = \pm 2, y_1 = 0)$; check: $x = 0$, $y = -3$; $y = 0$, $x = -5.6$, 1.6; (h) hyperbola $-x_1^2 + 4y_1^2 = 9$, $O_1(4, 2)$, asymptotes $x_1 = \pm 2y_1$, $V(x_1 = 0, y_1 = \pm\frac{3}{2})$; check: $x = 0$, $y = -\frac{1}{2}, \frac{9}{2}$; $y = 0$, $x = 1.3, 6.7$; (i) hyperbola $3x_1^2 - 2y_1^2 = 6$, $O_1(2, -3)$, asymptotes $y_1 = \pm\sqrt{\frac{3}{2}}\,x_1$; $V(x_1 = \pm\sqrt{2}, y_1 = 0)$; check: $x = 0$, $y = -4.7, -1.3$; $y = 0$, $x = 4.8$, -0.8
5. (a) $x_1y_1 = 15$, $O_1(-5, 7)$, asymptotes $x_1 = 0$, $y_1 = 0$, points (x_1, y_1): $(3, 5)$, $(5, 3)$, $(-3, -5)$, $(-5, -3)$; check: $x = 0$, $y = 10$; $y = 0$, $x = -\frac{50}{7}$; (b) $x_1y_1 = 300$, $O_1(40, 30)$, asymptotes $x_1 = 0$, $y_1 = 0$, points (x_1, y_1): $(10, 30)$, $(30, 10)$, $(-10, -30)$, $(-30, -10)$; check: $x = 0$, $y = \frac{45}{2}$; $y = 0$, $x = 30$; (c) $x_1y_1 = 2$, $O_1(-2, -3)$, asymptotes $x_1 = 0$, $y_1 = 0$, points (x_1, y_1): $(1, 2)$, $(2, 1)$, $(-1, -2)$, $(-2, -1)$; check: $x = 0$, $y = -2$; $y = 0$, $x = -\frac{4}{3}$; (d) pair of lines $x = -3$, $y = -6$; (e) $x_1y_1 = 300$, $O_1(-30, -40)$, asymptotes $x_1 = 0$, $y_1 = 0$, points (x_1, y_1): $(10, 30)$, $(30, 10)$, $(-10, -30)$, $(-30, -10)$; check: $x = 0$,

$y = -30$; $y = 0$, $x = -\frac{45}{2}$; (f) $x_1y_1 = -\frac{5}{2}$, $O_1(\frac{5}{2}, 3)$, asymptotes $x_1 = 0$, $y_1 = 0$, points (x_1, y_1): $(1, -\frac{5}{2})$, $(\frac{5}{2}, -1)$, $(-1, \frac{5}{2})$, $(-\frac{5}{2}, 1)$; check: $x = 0$, $y = 4$; $y = 0$, $x = \frac{20}{3}$

6. (a) demand, asymptotes $x = -2$, $p = 0$; $x \geqq 0$; use $(0, 4)$, $(2, 2)$, $(4, 1)$; (b) supply, asymptotes $x = -2$, $p = 5$; $x \geqq 0$; use $(0, 0)$, $(3, 3)$, $(8, 4)$; (c) demand, asymptotes $x = -2$, $p = 2$; $x \geqq 0$, $p \geqq 2$; use $(0, 7)$, $(3, 4)$, $(8, 3)$; (d) supply, asymptotes $x = -1$, $p = 5$; $x \geqq 0$, $1 \leqq p < 5$; use $(0, 1)$, $(1, 3)$, $(3, 4)$, $(7, \frac{9}{2})$

Problem Set 6-6

1. (a) $(0, 4)$, $(4, 0)$; (b) $(4, 0)$, $(-\frac{12}{5}, \frac{16}{5})$; (c) $(2, 3)$, $(-\frac{18}{5}, \frac{1}{5})$; (d) $(3, 2)$, $(3, 2)$
2. (a) $(\frac{16}{9}, \pm 3.6)$; (b) $(0, 0)$, $(\frac{9}{2}, \frac{9}{2})$; (c) $(4, 3)$, $(\frac{24}{5}, \frac{7}{5})$; (d) $(1, 1)$, $(-\frac{35}{61}, -\frac{19}{61})$
3. (a) $(1, 0)$, $(0, \frac{1}{2})$; (b) $(\sqrt{2}/2, -\sqrt{2}/4)$, $(-\sqrt{2}/2, \sqrt{2}/4)$; (c) none
4. (a) $(2, 12)$, $(\frac{13}{2}, \frac{39}{4})$; (b) $(1.79, 11.11)$, $(6.71, 8.64)$; (c) $(1, 0)$, $(\frac{9}{2}, \frac{7}{4})$; (d) $(0.61, 1.31)$, $(4.89, 3.44)$
5. (a) $(\frac{11}{3}, \frac{14}{9})$; (b) $(5.62, 3.24)$, $(3.38, -1.24)$; (c) $(1, 8)$, $(4, 2)$; (d) $(0.70, 11.40)$, $(-5.70, -1.40)$
6. (a) $(6, \frac{8}{3})$, (b) $(3.72, 6.72)$, (c) $(2, 22)$, (d) $(2.13, 23.6)$, (e) $(\frac{5}{2}, 4)$, (f) $(7.90, 8.63)$
7. (a) $(0, 0)$, $(5, 5)$; (b) $(0.44, 2)$, $(4.56, 2)$; (c) $(0, 0)$; (d) $(2, 2)$, $x = -\frac{14}{9}$ does not lead to a solution
8. $0 < x < 2$; $x = 1$, $F = 5$
9. $1 < x < 6$; $x = \frac{7}{2}$, $F = 6.8$
10. $(3, 2.2)$

Problem Set 7-1

1. A2, E3
2. $a - a = 0$, $a = a - 0$
3. E4, Th. 7-2
4. $b + (-a)$ is one solution: A3, A4; solution is unique: E2, E3, Th. 7-2
5. E2, Th. 7-1, definition of subtraction twice, E2, Cor. Th. 7-2
6. Cor. Th. 7-3, Th. 7-5, A3
7. Cor. Th. 7-3 used several times, Th. 7-5, A3, A5, A4

Problem Set 7-2

1. M2, E3
2. $a/a = 1$, $a/1 = a$
3. E4 twice to ac
4. E4, Th. 7-7
5. (a) M3, M5, M4, Th. 7-7; (b) 0
7. M4 and E6, M6, A5, Th. 7-10, definition of subtraction and A4
8. Th. 7-13, M2, M3, Th. 7-13; Th. 7-13, M3, part 1 of Th. 7-13, A5: $-(-c) = c$
9. Th. 7-13, M6, Th. 7-13; Th. 7-13, M6, Th. 7-13, A5
10. Cor. Th. 7-8, Th. 7-6, M5, M4, E4

Problem Set 7-3

1. Th. 7-14, O3
2. $a/b = c$ implies $a = bc$; suppose a is positive and b negative; if $c = 0$, Th. 7-10; if $c > 0$, Cor. 1, Th. 7-16, O1; if $c < 0$, Cor. 2, Th. 7-16, O1
3. $a + (x + y) = c$, O2
4. false
5. true, Th. 7-20 for $a + c > b + c$ and $(-c) = (-c)$, and properties of zero
6. hypothesis implies $a - b > 0$
7. product of a positive number and a negative number is negative
8. $a = b + x$, where xc is negative, $-xc$ positive
9. $a + x = b$, $c + y = d$, (x, y positive), O3, O2; example: $-3 < 2$ and $-4 < 3$ but $12 < 6$ is false
12. (a) to $a < b$ add $a = a$ and then $b = b$, (b) to $a < b$ add $2a = 2a$, $a + b = a + b$, $2b = 2b$

Problem Set 7-4

1. (a) $x > 3$, (b) $x > \frac{12}{5}$, (c) $x < 3$, (d) $x < -\frac{12}{5}$
2. (a) $x < \frac{3}{2}$, (b) $x < \frac{3}{2}$, (c) $x > -3$, (d) $x > \frac{3}{2}$
3. (a) $x > 3$ or $x < -\frac{1}{3}$, (b) $-2 < x < -\frac{6}{5}$, (c) $x > \frac{7}{2}$ or $x < \frac{3}{2}$, (d) $1 < x < 4$
4. $-\frac{1}{2} < x < \frac{1}{2}$ or $\frac{5}{2} < x < \frac{7}{2}$
5. (a) no x, (b) any x
6. $2 < x < 4$
7. (a) $1 < x < 3$, (b) $x > 4$ or $x < 1$
8. (a) $-1 < x < 2$, (b) $x > 2$ or $x < -2$
9. $x > -\frac{1}{2}$ or $x < -\frac{3}{2}$
10. $-4 < x < 2$

Problem Set 7-5

1. (a) every x, (b) no x, (c) every x, (d) no x, (e) no x, (f) every x
2. (a) $x > 2$ or $x < -3$, (b) $-3 < x < 2$, (c) $x > \frac{3}{2}$ or $x < -\frac{2}{3}$, (d) $-\frac{2}{3} < x < \frac{3}{2}$, (e) $\frac{5}{2} < x < 5$, (f) $x > 5$ or $x < \frac{5}{2}$
3. (a) $x > \frac{3}{2}$ or $x < -2$, (b) $x > 3$ or $x < -\frac{3}{2}$, (c) all x except $x = 3$, (d) no x, (e) $-3 < x < \frac{5}{6}$, (f) $x > \frac{5}{6}$ or $x < -3$
4. (a) $x > \sqrt{2} - 1$ or $x < -\sqrt{2} - 1$,
   (b) $(-\sqrt{5} - 1)/2 < x < (\sqrt{5} - 1)/2$,
   (c) $x < (3 - \sqrt{3})/2$ or $x > (3 + \sqrt{3})/2$,
   (d) $(-\sqrt{17} - 1)/4 < x < (\sqrt{17} - 1)/4$
5. (a) $x < \frac{1}{2}$, (b) $x > 1$, (c) $-1 < x < \frac{7}{3}$, (d) $x > 2$ or $x < -1$, (e) $x > 8$ or $x < \frac{4}{3}$, (f) $-6 < x < \frac{4}{5}$
6. (a) $x > 6$ or $-2 < x < 2$, (b) $x > 2$ or $x < -4$ or $-\frac{8}{3} < x < -2$

Problem Set 8-1

1. (d) $(10\ \ -6)$
2. (c) $(10\ 6\ 2)$
3. (c) $(-\frac{10}{3}\ \ -\frac{1}{3}\ \ \frac{1}{3})$
4. not conformable
5. both equal to $(u_1 + v_1 + w_1\ \ u_2 + v_2 + w_2)$
6. Th. 7-2 and Cor.
7. $k0 = 0$, $a + 0 = a$
8. $(-1)u = 0 - u$
9. (a) M6, (b) Def. 8-8, (c) Defs. 8-4, 8-8
10. yes, different, no

Problem Set 8-2

3. (a) $\frac{8}{5}$, $\frac{14}{5}$; (b) -1, 3; (c) $-\frac{6}{5}$, $-\frac{8}{5}$
4. $\mathbf{X} = -2\mathbf{U} + 5\mathbf{V}$, $\mathbf{Y} = 3\mathbf{U} - 2\mathbf{V}$, $\mathbf{U} = (2\mathbf{X} + 5\mathbf{Y})/11$, $\mathbf{V} = (3\mathbf{X} + 2\mathbf{Y})/11$
5. (a) $(17\mathbf{U} - 5\mathbf{V} + 16\mathbf{W})/3$, (b) $-3\mathbf{U} - \mathbf{W}$
6. (a) $\mathbf{X} = 2\mathbf{U} - 2\mathbf{V}$, $\mathbf{Y} = 3\mathbf{U} + 5\mathbf{V}$, coplanar;
   (b) $\mathbf{U} = (5\mathbf{X} + 2\mathbf{Y})/16$, $\mathbf{V} = (-3\mathbf{X} + 2\mathbf{Y})/16$
7. (a) $\begin{pmatrix} \frac{8}{3} \\ \frac{14}{3} \end{pmatrix}$, (b) $\begin{pmatrix} \frac{5}{2} \\ 5 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{9}{4} \\ \frac{11}{2} \end{pmatrix}$, (d) $\mathbf{U}$, (e) $\mathbf{V}$, (f) $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$, (g) $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
8. diagonals of a parallelogram bisect each other
9. (b) terminal points A, B, C are collinear; if $t < 0$, B is between A and C; if $t > 1$, A is between B and C
10. (a) $(\frac{1}{3}\ \ 0\ \ \frac{7}{3})$, C is between A and B;
    (b) $(-\frac{19}{3}\ \ 4\ \ -\frac{13}{3})$, B is between A and C;
    (c) $(\frac{11}{3}\ \ -2\ \ \frac{17}{3})$, A is between B and C
11. (a) $x = tx_1 + sx_2 + (1 - t - s)x_3$, $y = ty_1 + sy_2 + (1 - t - s)y_3$;
    (b) $t = \frac{1}{6}$, $s = \frac{1}{3}$, $1 - t - s = \frac{1}{2}$, interior; (c) $t = -\frac{1}{3} < 0$, exterior
12. (a) $x = tx_1 + sx_2 + (1 - t - s)x_3$ and similar equations involving y and z; (b) $t = \frac{1}{3}$, $s = \frac{1}{2}$, $1 - t - s = \frac{1}{6}$, interior
13. $t < 1$, since M must be between O and A; $\overrightarrow{OC} = \overrightarrow{OM} + \overrightarrow{ON}$; by similar triangles, $\overrightarrow{ON} = (1 - t)\mathbf{V}$

Problem Set 8-3

1. $\frac{7}{2}$, $-\frac{1}{2}$
2. (e) $\begin{pmatrix} -\frac{5}{6} & \frac{1}{3} \\ \frac{7}{6} & 2 \\ \frac{11}{6} & \frac{5}{3} \end{pmatrix}$, (f) $\begin{pmatrix} 2 & 7 \\ -7 & 6 \\ 4 & -11 \end{pmatrix}$
3. not conformable
4. use definitions of transpose and sum
5. $(a + b) + c = a + (b + c)$
6. if $a = b$, then $a + c = b + c$, and conversely
7. $k0 = 0$, $a + 0 = a$
8. $(-1)a = -a = 0 - a$
9. use $(k + l)a = ka + la$, $k(la) = (kl)a$, $k(a + b) = ka + kb$
10. (a) -10, (b) 0, (c) $5k$, (d) 0

Problem Set 8–4

1. (a) (-2), (b) not compatible, (c) (-5), (d) (15), (e) not compatible
2. $(\overrightarrow{OA}, \overrightarrow{OB})$, $(\overrightarrow{OC}, \overrightarrow{OD})$
3. $(\overrightarrow{OA}, \overrightarrow{OC})$, $(\overrightarrow{OA}, \overrightarrow{OD})$, $(\overrightarrow{OB}, \overrightarrow{OD})$
4. $12x + 16y + 10z - 4x^2 - 3y^2 - 5z^2$, $(0 \leqq x \leqq 3, 0 \leqq y \leqq \frac{16}{3}, 0 \leqq z \leqq 2)$
5. (a) $(4\ 7\ 0)$, (b) $(12\ 3)$, (c) not compatible
6. (a) $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$, (b) not compatible, (c) $\begin{pmatrix} -1 \\ 1 \\ 14 \end{pmatrix}$, (d) not compatible
7. (a) $\begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 7 & 14 \\ 5 & 8 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 6 \\ 5 & 8 \end{pmatrix}$, (d) same as (b)
   (e) $\begin{pmatrix} 8 & 3 \\ 5 & 0 \end{pmatrix}$, (f) $\begin{pmatrix} 3 & 6 \\ 5 & 5 \end{pmatrix}$
8. $\mathbf{AB} = \mathbf{AC} = \begin{pmatrix} 5 & -2 \\ 10 & -4 \end{pmatrix}$
10. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
11. (a) $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$, $(x\ y)\begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} = (7\ 2)$;

    (b) $\begin{pmatrix} 3 & 2 \\ -2 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$, $(x\ y)\begin{pmatrix} 3 & -2 & 5 \\ 2 & 1 & 1 \end{pmatrix} = (7\ 2\ 5)$;

    (c) $\begin{pmatrix} 3 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \end{pmatrix}$, $(x\ y\ z)\begin{pmatrix} 3 & -2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix} = (9\ 13)$;

    (d) $\begin{pmatrix} 3 & 2 & -1 \\ -2 & 1 & 3 \\ 5 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ 6 \end{pmatrix}$, $(x\ y\ z)\begin{pmatrix} 3 & -2 & 5 \\ 2 & 1 & 3 \\ -1 & 3 & -2 \end{pmatrix} = (9\ 13\ 6)$
12. $(5\ 15\ 3\ 7)\begin{pmatrix} 2 & 3 & 2 \\ 3 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & 4 & 2 \end{pmatrix}\begin{pmatrix} \$3 \\ \$1 \\ \$4 \end{pmatrix} = \$442$

Problem Set 8–5

1. (a) no, (b) yes
2. (a) any x, $y = x - 2$, (b) no (x, y)
3. the associative and distributive laws for real numbers show both products have the same form

4. the distributive law of multiplication and the associative law of addition for real numbers show both forms are equal

5. (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

6. products are (a) $\mathbf{I}_2$ (b) $\mathbf{I}_3$ (c) $\mathbf{I}_2$

7. $\mathbf{AB} = \mathbf{O}$, $\mathbf{BA} = \mathbf{B}$

9. $\mathbf{BA} = \begin{pmatrix} 6 & 12 \\ -3 & -6 \end{pmatrix}$

10. (a) if $\mathbf{A} = \mathbf{B}$, rule of substitution shows $\mathbf{AC} = \mathbf{BC}$,
(b) $\mathbf{AC} = \begin{pmatrix} 10 & 15 \\ -6 & -9 \end{pmatrix} = \mathbf{BC}$,
(c) $\mathbf{CA} = \begin{pmatrix} 5 & -2 \\ 10 & -4 \end{pmatrix} = \mathbf{CB}$, $\mathbf{A} \neq \mathbf{B}$, $\mathbf{C}$ does not have an inverse

11. $\mathbf{ACC}^{-1} = \mathbf{BCC}^{-1}$ implies $\mathbf{A} = \mathbf{B}$, $\mathbf{C}^{-1}\mathbf{CA} = \mathbf{C}^{-1}\mathbf{CB}$ implies $\mathbf{A} = \mathbf{B}$

Problem Set 8-6

1. (a) $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$, (b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, (c) $\begin{pmatrix} \frac{21}{16} \\ \frac{3}{16} \end{pmatrix}$, (d) $\begin{pmatrix} \frac{1}{13} \\ \frac{7}{13} \end{pmatrix}$

2. (a) $\begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$, (b) $\begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$, (c) $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, (d) $\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

5. (a) $(x, 22x - 41, 17x - 36)$, $([z + 36]/17, [22z + 95]/17, z)$;
(b) $(x, -2x + \frac{13}{7}, x - \frac{2}{7})$, $(z + \frac{2}{7}, -2z + \frac{9}{7}, z)$

6. (a) $x = -(z + 5)/2$, $y = (z - 4)/2$;
(b) $x = -(2y + 9)/2$, $z = 2y + 4$

7. (a) $k = -1$, $x = 7 - z$, $y = 2z - 5$;
(b) $k = \frac{7}{5}$, $x = (9 - 5z)/5$, $y = (10z - 1)/5$

8. $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

10. write the corresponding equations and solve by Cramer's Rule

Problem Set 8-7

1. intersection (3, 5), (0, 0) is in the (+ −) region
2. (a) (− − −), (b) (− + −)
3. (a) interior of triangle (0, 0), (2, 0), (0, 3);
(b) unbounded region not containing O; $(\frac{5}{2}, 0)$, $(0, \frac{5}{4})$;
(c) no region; (d) interior of triangle (2, 3), (−2, 3), (2, −3)
4. (a) unbounded region containing O; (3, 0), $(-\frac{3}{4}, \frac{5}{4})$;
(b) interior of angle, vertex (3, 0), containing such points as (5, −1)

5. (a) interior of triangle $(0, 2)$, $(\frac{9}{5}, \frac{4}{5})$, $(6, 5)$;
   (b) interior of triangle $(1, 3)$, $(5, 1)$, $(3, 5)$
6. (a) unbounded quadrangular region; $(0, 3)$, $(\frac{6}{5}, \frac{6}{5})$, $(3, 0)$;
   (b) interior of quadrilateral $(0, 0)$, $(2, 0)$, $(2.4, 0.8)$, $(0, 2)$;
   (c) no region
7. (a) $5x = 6 - 2u + 3v$, $5y = 6 + 3u - 2v$, $x, y, u, v \geqq 0$; unbounded quadrangular region with vertices $(0, 3)$, $(0, 0)$, $(3, 0)$;
   (b) $10x = 24 - 2u - v$, $10y = 8 + u - 2v$, $x, y, u, v \geqq 0$; quadrilateral with vertices $(0, 0)$, $(12, 0)$, $(8, 8)$, $(0, 4)$;
   (c) $5x = -2 - 2u - v$, $5y = 11 + u - 2v$, $x, y, u, v \geqq 0$, no region
8. (a) max. 12, min. -3; (b) no max., min. $\frac{13}{4}$; (c) no max., no min.;
   (d) no max., min. 18; (e) max. 68, min. 7; (f) max. 43, min. 19
9. max. 5 at $(\frac{3}{2}, -\frac{3}{2})$, min. -14 at $(-7, 5)$
10. (a) triangular pyramid $(0, 0, 0)$, $(2, 0, 2)$, $(0, 0, 4)$, $(0, 2, 2)$;
    (b) quadrilateral pyramid $(0, 0, 0)$, $(4, 0, 0)$, $(2, 0, 2)$, $(0, 3. 1)$, $(0, 4, 0)$;
    (c) triangular pyramid $(2, 3, 4)$, $(2, 3, 0)$, $(2, 0, 4)$, $(0, 3, 4)$
11. (a) max. 12, min. 0; (b) max. 8, min. -8; (c) max. 14, min. -4

Problem Set 8-8

1. (a) $\frac{32}{3}$ lb pork, min. $Q = \$6.40$; (b) $0 \leqq x$ lb beef $\leqq \frac{10}{3}$.
   pork: $y = (160 - 18x)/15$; min. $Q = \$8$
2. min. $Q = 66$ at $(3, 0, 4)$
3. (b) at B, $P = \$300$; at F, $P = 250 + 25t < 300$
4. $t = 2$, $0 \leqq y \leqq 100$, number of turkeys $= 200 - y$, max. $P = \$400$;
   $t = 3$, all turkeys, max. $P = \$600$
5. $t = 2$, $x = 0$, $150 \leqq y \leqq 200$, balance turkeys, max. $P = \$400$;
   $t = 3$, $x = 0$, $y = 150$, 50 turkeys, max. $P = \$450$
6. $t = 1$, $y = 50$, $100 \leqq x \leqq 150$, $150 - x$ turkeys, max. $P = \$250$
   $t = 2$, $x = 100$, $y = 50$, 50 turkeys, max. $P = \$300$
7. (b) consider the family of lines $2x + y = P$
8. (a) 16 for $(2, 4)$, (b) 16 for $(4, 2)$
9. max. $F = \frac{116}{11}$ at $(\frac{28}{11}, \frac{8}{11})$
10. max. $F = 38$ at $(6, 5)$, min. $F = 6$ at $(2, 0)$

Problem Set 9-1

1. (a) $1/x^5$, (b) $1/x$, (c) 1, (d) $1/x^4$, (e) $1/x^6$, (f) $1/x^6$,
   (g) x^6, (h) $1/x^3$, (i) x^6/y^6, (j) $1/(x^6y^6)$, (k) x^4y^6
4. (a) $(27^2)^{1/3} = (27^{1/3})^2 = 9$, (b) $(16^3)^{1/4} = (16^{1/4})^3 = 8$,
   (c) $\left[\left(\frac{3}{4}\right)^3\right]^{1/2} = \left[\left(\frac{3}{4}\right)^{1/2}\right]^3 = \frac{3\sqrt{3}}{8}$, (d) $(a^6b^2)^{1/3} = a^2b^{2/3}$,
   (e) $(a^3/b^2)^{1/2} = a^{3/2}/b$, (f) $a\sqrt{a}$, (g) $ab^2\sqrt{ab}$, (h) $b^2\sqrt[3]{a^2}$
5. (a) $x^{5/6}$, (b) $1/x^{1/6}$, (c) $1/x^{1/2}$, (d) $x^{1/2}$, (e) $2/(3x^{1/3})$,
   (f) $-1/(3x^{4/3})$

8. use Th. 9–2 and if $x > 1$, $1/x < 1$
10. (0.9, 7.7) or (0.88, 7.64)

Problem Set 9–2

1. (a) 3, −4; (b) 16, $\frac{1}{8}$; (c) 3, −3; (d) 10000, 0.0001
2. (c) $\log x = (\log A + 2 \log B - \log C)/3$
4. (c) 3.6271, 3.8858, 8.4583 − 10; (d) 4.9443, 1.9441, 7.9444 − 10; (e) 0.8384, 9.8832 − 10, 0.0099
5. (a) 189.0, 0.01890, 0.001890; (b) 7870, 0.7870, 0.07870; (c) 1.965, 0.001965, 0.1965; (d) 26.77, 0.2677, 0.002677; (e) 522.4, 0.05224, 0.5224

Problem Set 9–3

1. (a) 92380000, (b) 1122, (c) 1.067, (d) 0.05965, (e) 0.7838, (f) 0.01410
2. (a) 9.268, (b) 1.123, (c) 47.88, (d) 5.802, (e) 0.7834, (f) 0.1162
3. (a) 5.175, (b) 2.991, (c) 0.9002, (d) −0.5314, (e) 4.079, (f) −1.325
4. (a) 3.013, (b) 0.4373, (c) 17.56, (d) 1.847, (e) 2.590, (f) 0.7943
5. (a) $Y = 0.6021 + x/3$; (b) 8.618, 86.18; (c) 1.194, 2.097
6. (a) $Y = 1.4771 - x/2$; (b) 9.486, 0.9486; (c) 0.9542, 3.556
7. (a) $P = 1.3010 - 3X/2$, $-0.6 < X < 0.6$; (b) 56.56, 3.849; (c) 1.587, 0.6300
8. (a) $Y = 4.3010 + 0.32X$, (b) 40400, (c) 35480, (d) 2.55 yr
10. (a) 1.292, (b) −3, (c) 0.3219, (d) 0, (e) 1.299, (f) 0.7519

Problem Set 9–4

1. (a) \$1537.98, (b) \$1552.24, (c) \$1559.56
2. (a) \$376.89, (b) \$370.17, (c) \$368.64
3. \$562.11
4. \$712.82
5. \$241.26, \$241.41
6. \$175.05
7. (a) 29 half-years, \$204.6; (b) 56 quarters, \$200.5
8. 30 yr, \$302
9. (a) 2.9(2.91)%, (b) 3.7(3.65)%
10. (a) 1.330, (b) 18.7 yr
11. (a) 3.72%, (b) 3.7%
12. (a) \$8348.06, (b) \$8345.62
13. (a) \$1029.56, (b) \$1030.00, \$1029.13
14. (a) \$658.58, (b) \$1022.75
15. (a) \$2876.46, (b) \$3671.18
16. \$3827.47
17. \$1033.75, \$1319.35, \$1683.87

Problem Set 10–1

1. (a) 79, 820; (c) $\frac{41}{6}$, $\frac{220}{3}$; (d) 13, $\frac{400}{3}$; (f) −42, −80
2. (a) 53, 306; (b) 108, 636; (c) −0.9, −0.9; (d) $c = 1$: $-\frac{7}{3}$, −6; $c = 2$: $\frac{26}{3}$, 60
3. 5576
4. 4315

7. (a) $n(3n + 1)/2$
10. verify for $n = 1$; assume $x^{2k} - 1 = (x + 1)Q(x)$ and write $x^{2k+2} - 1 = (x^{2k+2} - x^{2k}) + (x^{2k} - 1)$
11. \$574
12. $d = 12.3\%$, $r = 13.2\%$
14. $r = 11.37\%$, $d = 10.43\%$
15. $r = 34.3\%$, $d = 31.2\%$

Problem Set 10–2

1. (a) $\frac{1}{32}$, $\frac{1023}{32}$; (d) -120, -64.7
2. (a) ± 4; (b) -32; (c) $c = 0$: $\frac{3}{4}$; $c = \frac{9}{16}$; 3; (d) $\frac{8}{3}$
3. 262142
4. four
6. 27, $\frac{27}{5}$
7. 5 ft
8. (a) $\frac{21}{55}$, (b) $\frac{105}{37}$, (c) $\frac{2447}{990}$
9. (a) 4.672, (b) 27, (c) $32\sqrt{2} = 45.25$, (d) 8.221
11. 1.02470
12. 1.10490, 0.90506
13. 7.4¢
14. \$579.55
15. \$138.04
16. \$1457.84

Problem Set 10–3

1. (a) \$3259.86, \$698.44; (b) \$581.90, \$242.81; (c) \$16387.9, \$9007.35; (d) \$5558.52, \$986.94
2. (a) \$6.15, (b) \$106.43, (c) \$1.74, (d) \$9.95
3. \$1371.13
4. \$61.75
5. (a) \$853.02, (b) \$3159.89, (c) \$3333.33
6. \$12887.8
7. \$47.45
8. $14\frac{1}{2}$ yr, no; without 29th payment \$4085.63
9. 52 mo, \$20.32
10. 21.5% compounded monthly
11. 11% compounded monthly
12. (a) $d = 8.1\%$, (b) 8.5% compounded monthly
13. 4.4%

Problem Set 10–4

1. (a) \$112.55, (b) \$89.32
2. \$1045.59
3. (a) \$1045.57, (b) \$1408.23
4. (a) $\$20a_{\overline{10}|}$, (b) $\$20s_{\overline{10}|}$, (c) $\$20(a_{\overline{13}|} - a_{\overline{3}|})$, (d) $\$20(s_{\overline{17}|} - s_{\overline{7}|})$, (e) $\$20(s_{\overline{5}|} + a_{\overline{5}|})$
5. 30 mo, \$1003.55
8. \$605.21
9. (a) $a_{\overline{120}|}[1 + (1.005)^{-120}]$, $s_{\overline{120}|}[1 + (1.005)^{120}]$ or $a_{\overline{200}|} + a_{\overline{40}|}(1.005)^{-200}$, $s_{\overline{200}|}(1.005)^{40} + s_{\overline{40}|}$; (b) $a_{\overline{36}|}[1 + (1.03)^{-36}]$, $s_{\overline{36}|}[1 + (1.03)^{36}]$; (c) $a_{\overline{50}|}[1 + (1.06)^{-50}]$, $s_{\overline{50}|}[1 + (1.06)^{50}]$
10. 308.756, 15.8131
11. (a) \$2710.42, (b) \$3057.93
12. \$1535.85
13. (a) \$43.87, (b) \$512.45
14. (a) \$922.51, (b) \$6084.8
15. (a) 20, (b) \$46.32, (c) \$437.22
16. (a) 17, \$646.7; (b) \$6347.4
17. 79 quarters
18. 17 years without a final payment, \$2600
20. 1.16129

INDEX

Absolute value, 11, 129, 138
  and inequalities, 176
Accumulated amount, 262, 312
  of an annuity, 285
  at simple discount, 82
  at simple interest, 80
Activity, analysis, 238 ff.
  matrix, 209
Addition, 1, 4, 7, 163, 187
  axioms of, 164
  of decimal fractions, 41
  of equals, 9, 165
  of inequalities, 173
  of matrices, 200
  as permissible operation, 65
  of vectors, 190
Additive inverse, 8, 165
  of a matrix, 201
  of a vector, 189
Affine plane, 141
Algebraic expression, 5
Algebraic fraction, 38
  addition of, 62
  simplification of, 61
Algebraic operations with square roots, 131
Amount remaining due, 296
Annuity, 279, 284
  deferred, 293
  due, 292
  formulas, 284
  forwarded, 284
  problems, 287
  tables, 286, 320 ff.
Arithmetic, average, 272
  progression, 271
Associative law, for addition, 7, 164
  of matrices, 201
  for multiplication, 22, 167
    of matrices, 212
  of vectors, 189
Asymptotes of a hyperbola, 153
Auxiliary circle of an ellipse, 151
Average, arithmetic, 272
  geometric, 278
  time, 273, 274
Axiom systems, 163
Axioms, addition, 186
  equality, 186
  least upper bound, 164, 185
  linear measure, 13
  multiplication, 186
Axis of symmetry for parabola, 142

Base, in percentage, 48
  of system of logarithms, 251
Between, 14, 195
Bonds, 291
Boundary, 226
Bounded above, 185
Binomial, 29
Binomial Theorem, 281, 291
Budget equation, 95, 121

Cancellation, for addition, 9, 165
  of matrices, 202
  for multiplication, 23, 168
    of matrices, 215
  of vectors, 190
Characteristic, 253, 255
Circle, 151
Closure, for addition, 7, 164
  of matrices, 201
  for multiplication, 22, 167
    of matrices, 211
  of vectors, 189
Cofactor, 111
Column vector, 187, 199
Commission, 48
Common fractions, 1
Common logarithms, 251
  tables of, 253, 310
Commutative law, for addition, 7, 164
  failure for addition and square root, 132
  failure for matrices, 211, 213, 215
  of matrices, 201
  for multiplication, 22, 167
  for multiplication and square root, 132
  of vectors, 189
Compatible, 207
Complementary commodities, 121
Complete quotient, 40

Completing the real number system, 184
Completing the square, 134, 142, 154
Complex fractions, 38
  numbers, 2, 136
Components of a vector, 187
Compound, amount, 269
  interest, 261, 285
Computer's rule, 45
Conditional equations, 64
Conformable vectors, 188
Conic, 158
Constant, 2
  function, 85
Constraints, 234, 237
Continuity, axiom of, 62
Convex, 226, 228
Coordinate, 13
  on a line, 86
  in a plane, 86
Coordinate axes, 85, 120
Coordinate planes, 120
Corner, method, 234, 237
  points, 231 ff.
Cost, 149
Cramer's rule, 116

Decimal approximation, 45
  to $\sqrt{n}$, 130, 306
Decimal fractions, 1, 40, 164
  addition of, 41
  conversion to rational fractions, 42
  division of, 41
  multiplication of, 41
  nonterminating, 41
  periodic, 41, 280
Deferred annuity, 293, 294
Degree of a term, 29
Demand curve, equilateral hyperbola, 156
  line, 108
Demand laws, 93, 108, 147, 156, 250, 259
Dependent equations, 47, 100, 122 ff.
Determinants, 145, 203
  second-order, 99
  third-order, 111
Diagonalization method, 220 ff.
Diet problem, 235
Direct proportion, 80
Directed distance, 13
Discount, banker's, 81
  rate, 82
  simple, 80
  trade, 50
Discounts, successive, 77
Distance formula, 301
Distributive law, 23, 169
  for matrices, 213
Division, 4, 32, 168, 169
  of decimal fractions, 41 ff.
  of estates, 72
  as inverse of multiplication, 33
  of a line segment, 71
  of polynomials, 56
  as successive subtraction, 39
Division process for $\sqrt{n}$, 130, 305
Domain, of a function, 84
  of a variable, 3
Double interpolation, 290

Elimination, 98, 115
  by addition, 98
  method, 237
Ellipse, 151, 154
  auxiliary circle of, 151
Equality, 4, 6, 163, 187
  axioms of, 164
  of matrices, 200
  of vectors, 188
Equation, 5
  linear, 64
  quadratic, 129
Equation of a line, 86
  determined by two conditions, 103
  general form, 88
  point-slope form, 87
  slope-y-intercept form, 87
Equation of a plane, 301
Equation of value, 266, 287
Equations, dependent, 100
  inconsistent, 100
  involving absolute values, 138
  involving square roots, 136
Equilateral hyperbola, 153, 155, 178
Equivalent equations, 65
Equivalent fractions, 34
Equivalent pairs of equations, 97
Equivalent rates for income taxes, 78
Equivalent rates for successive discounts, 78
Everywhere dense, 184

Exponential equation, 259
Exponents, 28
  integral, 245
  irrational, 248
  negative, 245
  positive, 28, 245
  rational, 246
  zero, 245

Factoring, 52 ff., 133
  polynomials, 59 ff.
  by trial, 54
Factor Theorem, 58
Failure of laws of multiplication for matrices, 211
  cancellation, 215
  commutative, 215
  zero, 215
Formulas, 5, 68
Fractions, 32
  algebraic, 38
  complex, 38
  equivalent, 34
  improper, 40
Fractions, addition, 36 ff.
  change of form, 36 ff.
  division, 36 ff.
  multiplication, 36 ff.
  operations with, 15, 36 ff.
Function of one variable, 84
Functional evaluation, 85

Geometric average, 278
Geometric progression, 276, 285
Geometric vector, 192
Graduated income tax, 78
Graph, of equation, 141
  of linear equation, 85 ff.
Greater than, 4, 10, 163, 171

Half-planes, 226
Half-spaces, 228
Homogeneous linear equations, 125
Homogeneous polynomial, 29
Hyperbola, 153

Identities, 64
Identity function, 85
Identity matrix, 213
Imaginary numbers, 136
Incomplete induction, 262
Inconsistent equations, 100, 122 ff.
Inequalities, 7, 163, 171
  algebraic operations with, 172
  and equivalent equalities, 229
  involving absolute values, 176
  involving fractions, 177
  linear, 175
  quadratic, 180
  and squaring, 174
Infinite geometric progression, 279
Insertion of parentheses, 16, 27, 65
Integers, 1
Intercepts, for a line, 87 ff.
  for a parabola, 141
Interest, compound, 261, 285
  simple, 80
Interest bearing notes, 266
Interior of triangle, 196
Interpolation, 254, 287
  double, 263
  for interest rate, 264
  for time, 283
Intersection, of line and conic, 158
  of two lines, 100
Intrinsic value of a logarithm, 254
Inverse, additive, 8, 165
  of a matrix, 217 ff.
  multiplicative, 22, 168
  proportion, 153
Irrational exponents, 248
Irrational numbers, 1, 129
Irrationality of $\sqrt{2}$, 129
Irregular payment, 287, 296

Law of addition of equals, 9
Law of cancellation, for addition, 9, 165
  of matrices, 202
  of matrices (failure), 215
  for multiplication, 9, 23, 168
  of vectors, 190
Law of multiplication by a number, 23
Law of Trichotomy, 11, 171
Laws, of equality, 6
  of exponents, 28, 245, 248
  of inequality, 248 ff.
  of logarithms, 251
Least common denominator, 62
Least upper bound, 185, 248
Least upper bound axiom, 164

Legend, 2
Length, 13
Less than, 4, 10, 163, 171
Levers, 73
Linear function, 85
Linear inequalities, 175, 226, 228
Linear interpolation, 254
Linear laws, of demand, 93
  of supply, 93
Linear measure, axiom of, 13
Linear programming, 234, 242
Linearly dependent, 126
Logarithmic form of an equation, 252, 256
Logarithmic skeleton, 256
Logarithms, 95, 251
  intrinsic value, 254
  numerical calculations with, 256

Mantissa, 253
Market equilibrium, 108
  and additive tax, 109
Markup and markdown, 49
Mathematical induction, 269
Matrices and linear equations, 217
Matrix, elements of, 199
  multiplication, 207, 211
  order of, 199
  square, 199
Matrix algebra, 187
Maximum and minimum of a linear function, 230
Minor, 111
Mixture, 73
Monomial, 29
Multiplication, 4, 163, 187
  axioms of, 167
  of decimal fractions, 41 ff.
  of equals, 168
  of fractions, 36
  of inequalities, 173
  by an integer, 18
  laws of, 22, 168
  of matrices, 207
  as permissible operation, 65
  of polynomials, 29
  scalar, for matrices, 202
    for vectors, 190
  and special products, 30
  of vectors, 204 ff.
  by zero, 23, 169
Multiplicative inverse, 22, 32, 168
  of a matrix, 214

Negative, of an equality, 173
  of a matrix, 201
  of a number, 8
  of a vector, 189
Negative numbers, 2, 10, 164
Normal to a plane, 302
Null set, 97
Number, reciprocal of, 32, 168
Number field, 164, 186
Number scale, 12, 13, 85
Numbers, 1 ff.
  counting, 1
  complex, 136
  imaginary, 136
  irrational, 1, 129
  natural, 1
  rational, 1
Numerical calculations, 256

Objective function, 231, 234
One, 22
  axiom of, 168
Operations with numbers, 2, 4
Order, axioms of, 171
Origin, 120

Parabola, 141, 154
  axis of symmetry of, 142
  as demand, supply, total cost, revenue curves, 147
  intercepts of, 141
  through three points, 144
  vertex of, 142
Parallel line and plane, 123
Parallel lines, 126
Parallel planes, 122, 303
Parallelogram law, 192
Parentheses, 8, 16
Partial difference, 254
Partial quotient, 40
Per cent, 48
Percentage, 48
Perimeter, 71
Permissible operations, 65, 97, 136
Perpendicular vectors, 205
Perpetuity, 286
Plane, affine, 141
  equation of, 301

Euclidean, 141
separation postulate, 226
Polynomial, 29
Polynomials, division of, 56
Positive, 7, 171
Positive integers, 1
Positive integral exponents, 28, 35
Positive numbers, 10, 26, 164
Present value, of an annuity, 285, 324
at compound interest, 262, 316
at simple discount, 82
at simple interest, 82
Price as a vector, 206
Principal, 80
Principal root, 246
Product of reciprocals, 32
Profit and loss, 49, 160
Progression, arithmetic, 271
geometric, 276
Properties of (−1), 170
Proportion, direct, 80
inverse, 153
Proportional parts, 254
Prospective method, 292

Quadratic equation, 129 ff.
algebraic solution, 133
formula, 135
function, 141
inequalities, 180
Quotient, 34
complete, 40
partial, 40

Radicals, 246
Rate, 48
Rates, equivalent, 77
Range of a function, 84
Rational exponents, 246
Rational fractions and decimal fractions, 42 ff.
Rational numbers, 1, 184
Ray, 14
Real number, 2, 163, 168
Real number system, 1, 163, 184
Reciprocal, 32, 168
of reciprocal, 32, 168
Reflection, 13
and rule of sign, 15
Reflexive law of equality, 6, 164
Relations between numbers, 2, 4
Remainder, 39 ff.
Remainder Theorem, 58
Removal of parentheses, 16, 19, 27, 65
Rent, 284
Repeating decimal fractions, 41, 280
Retrospective method, 292
Revenue, 149, 159
as a vector, 206
Root of a negative number, 247
Round off, 45
in addition and subtraction, 46
in multiplication and division, 46
Row vector, 187, 199
Rules of sign, for addition and subtraction, 14 ff., 166
for division, 34
for multiplication, 25, 170
Rule of substitution, 7, 164

Scalar multiplication, for matrices, 202
for vectors, 190
Second-order determinants, 99
Segment, 14
Separates, 226, 228
Sign of position, 111
Significant figures, 45
Simple discount, 80, 81, 273
and interest rates, 83, 274
Simple interest, 80, 265
exact, 81
ordinary, 81
Simultaneous linear equations, 97 ff.
three variables, 115, 119
two variables, 97
worded problems, 104
Simultaneous linear inequalities, 228
Slack variable, 229, 234
Slope, 86 ff.
relative, 90
Solution set, 3, 5, 64, 66, 97
Special products, 30, 54
Square matrix, 202
Square of a number and inequalities, 174
Square root, 4, 129
algebraic operations with, 131
division process, 130, 305
equations involving, 136
of a product and a quotient, 131
table of, 309
Squares, table of, 309

Subsidy, 109
Substitution, rule of, 7, 164
Subtraction, 1, 4, 9, 164, 166
  of matrices, 201
  of vectors, 189
Successive discounts, 77
  and equivalent rate, 77
Successive elimination, 234
Sum, of matrices, 200
  of vectors, 188
Supply, law of, 147
  linear, 93, 108
Symbols, to represent numbers, 2
  operations, 4
  relations, 4
System of linear equations, in matrix form, 208
  solution, 83 ff., 220 ff.
Symmetric law of equality, 6, 164

Tables, accumulated amount, 312
  accumulated value of annuity, 320
  common logarithms, 310
  present value, 316
  present value of annuity, 324
  squares and square roots, 309
Tabular difference, 254
Taxes, 51
  additive, 109
Theorem, 163
Third-order determinant, 111, 116, 145
Total cost, 149, 159
Traces of a plane, 120, 302
Trade discount, 50
Transitive law, for equality, 6, 164
  for inequality, 173
Translation of axes, 154
Transpose, of a matrix, 200
  of a vector, 187, 205
Transposition, 67
Trichotomy, axiom of, 171
  law of, 11, 172
True sentences, 64

Unit matrices, 213
Uniform motion, 74, 90
Unknown, 3

Variable, 3
  domain of, 3
Vector, column, 187
  geometric, 192
  multiplication, 204 ff.
  row, 187
  transpose of, 188
Vertex of a parabola, 142

Zero, 2, 8, 23, 165
  and one, 24, 25
  axiom of, 165
  properties of, 169
  matrix, 201, 214
  vector, 189

TOSHO INSATSU
PRINTING CO., LTD.
TOKYO, JAPAN